A Guide to Boas (1983)

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Search on "stuck" to find problems I couldn't solve!

Chapter 1

Infinite Series, Power Series

1.1 The geometric series

1.1.1 Problems

Chapter 1 Problem 1.1

Find the height of the ball after the 10th bounce, which is the 11th term of a geometric series, $r = 2/3$. The eleventh term is

$$r^{10} = \left(\frac{2}{3}\right)^{10} = \frac{1023}{81 \cdot 81 \cdot 9} \approx \frac{1}{59}$$

The distance travelled when it touches the ground the 10th time is $S_{n=10}$

$$S_{10} = \frac{a(1 - r^{10})}{1 - r} \approx 3\,a\frac{58}{59}$$

Compared to the total distance travelled,

$$\frac{S_n}{S} = 1 - r^n \approx \frac{58}{59}$$

Chapter 1 Problem 1.2 Prove the results (1.4).

This is surprisingly easy! Following her hint

$$S_n - S_n\, r = (1 - r)\, S_n$$

Now simply write out the sum of the terms on the RHS, and find that all but the first and last terms cancel

$$(1 - r)\, S_n = a - a\, r^n$$

so

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

which is (1.4).

Chapter 1 Problems 1.3 to 1.11 This is quite amazing. She claims that any fraction whose decimal equivalent does not terminate can be written as a sum of an infinite geometric series or as a repeating decimal. Note that this does not include the irrationals, but does apply to all rationals.

3.

$$0.55555.... = 0.5 + 0.5 \times \frac{1}{10} + 0.5 \times \frac{1}{10^2} + 0.5 \times \frac{1}{10^3} +$$

So $0.555... = S$ where $a = 5/10$ and $r = 1/10$,

$$S = \frac{5}{10}\,\frac{1}{1 - 1/10} = \frac{5}{10}\,\frac{10}{9} = \frac{5}{9}$$

4.

$$0.8181... = \frac{81}{100} + \frac{81}{100}\,\frac{1}{100} + \frac{81}{100}\,\frac{1}{100^2}...$$

So

$$0.8181... = S = \frac{81}{100} \times \frac{1}{1 - 1/100} = \frac{81}{99}$$

5.

$$0.58333... = \frac{58}{100} + \frac{3}{900} = \frac{87}{150}$$

The remaining problems through 11 do not involve any further concepts.

1.2 Definitions and Notations

1.2.1 Problems

There's not too much to the problems.

Chapter 1 Problem 2.6

For this one t's probably useful to first make the simplification

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} = \sum_{n=1}^{\infty} \frac{(n!)}{(2n \times (2n-1) \times ...(2n-(n-1))}$$

1.3 Applications of series

She clarifies that the "elementary functions" are: powers and roots, trignometric and inverse trignometric, exponentials and logs, and combinations of these.

An application will be the representation of functions that cannot be represented by combinations of elementary functions.

No problems for this section.

1.4 Convergent and divergent series

Nice example of the danger of manipulating series!

$$S = 1 + 2 + 4 + 8 + ...$$

and

$$2\,S = 2 + 4 + 8 + ...$$

So it might appear that $S - 2S = 1$, so $S = -1!$, which is nonsense.

1.4.1 Problems

Here she gives a careful definition of convergent in terms of ϵ and N.

For all the problems we want to prove convergence. That is, we find the N such that $|S - S_n| < \epsilon$ for all $n \geq N$.

These are quite fun problems, and feel more mathematical than most physical science problems.

Chapter 1 Problem 4.1

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

We recognize this as a geometric series with $a = 1/2$ and $r = 1/2$, so

$$|S - S_n| = \left| \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} \right| = \frac{r^n}{1-r} = \frac{1}{2^n}$$

Want to find condition such that

$$\frac{1}{2^n} < \epsilon$$

for a given ϵ. We simply solve the inequality for n,

$$n > \frac{-\ln(\epsilon)}{\ln(2)}$$

Chapter 1 Problem 4.2

This is essentially the same as the previous problem.

Chapter 1 Problem 4.3

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

So if we truncate the series at $n = N$, the remainder not accounted for is

$$S - S_N = \sum_{n=N+1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

We note that all terms cancel accept the first term

$$\frac{1}{N+1}$$

and the "last term",

$$\lim_{n\to\infty} -\frac{1}{n+1} \to 0.$$

So

$$S - S_N = \frac{1}{N+1}$$

And we want

$$S - S_N = \frac{1}{N+1} < \epsilon$$

for some specified ϵ. Solving for N,

$$N > \frac{1}{\epsilon} - 1$$

Chapter 1 Problem 4.4

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

Again she basically gives us the answer by noting that

$$\frac{1}{n!} < \frac{1}{2^n}, \quad n > 3$$

We note that

$$S - S_N = \sum_{n=N+1}^{\infty} \frac{1}{n!} < \sum_{n=N+1}^{\infty} \frac{1}{2^n}, \quad N > 2$$

The far RHS is a geometric series with first term

$$a = \frac{1}{2^{N+1}}$$

and $r = 1/2$, and therefore sum

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^{N+1}} \frac{1}{1 - 1/2} = \frac{1}{2^N}$$

So

$$S - S_N < \frac{1}{2^N} < \epsilon$$

when

$$N > \frac{-\ln(\epsilon)}{\ln(2)}$$

Chapter 1 Problem 4.5

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

Again she basically gives us the answer by noting that

$$\frac{1}{n\,2^n} < \frac{1}{2^n}, \quad n > 1$$

So it reduces to the previous problem,

$$S - S_N < \frac{1}{2^N} < \epsilon$$

when

$$N > \frac{-\ln(\epsilon)}{\ln(2)}$$

Chapter 1 Problem 4.6

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$$

Again she basically gives us the answer by noting that

$$\frac{1}{2^n + 3^n} < \frac{1}{2^n},$$

So it reduces to the previous problem,

$$S - S_N < \frac{1}{2^N} < \epsilon$$

when

$$N > \frac{-\ln(\epsilon)}{\ln(2)}$$

Chapter 1 Problem 4.7

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Again she basically gives us the answer by noting that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

and furthermore,

$$\frac{1}{(n+1)^2} < \frac{1}{(n+1)\,n}$$

So if we truncate the series at $n = N$,

$$S - S_N = \sum_{n=N+1}^{\infty} \frac{1}{n^2} = \sum_{n=N}^{\infty} \frac{1}{(n+1)^2} < \sum_{n=N}^{\infty} \frac{1}{n\,(n+1)}$$

From **Chapter 1 Problem 4.3**, we note

$$\sum_{n=N}^{\infty} \frac{1}{n\,(n+1)} = \sum_{n=N}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{N}$$

Thus

$$S - S_N < \frac{1}{N} < \epsilon$$

when

$$N > \frac{1}{\epsilon}$$

Chapter 1 Problem 4.8

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

As a challenge, I tried to ignore her hints for this one. I noted that

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \sum_{n=2}^{\infty} \frac{1}{(n-1)\,(n+1)} = \sum_{n'=1}^{\infty} \frac{1}{n'\,(n'+2)}$$

Truncating the sum at $n' = N$, the partial sum has a residual,

$$S - S_N = \sum_{n'=N+1}^{\infty} \frac{1}{n'\,(n'+2)} < \sum_{n'=N+1}^{\infty} \frac{1}{n'\,(n'+1)} = \frac{1}{N+1}$$

where we used the result from **Chapter 1 Problem 4.3**. Thus

$$|S - S_N| < \frac{1}{N+1} < \epsilon$$

when

$$N > \frac{1}{\epsilon} - 1$$

1.5 Testing series for convergence: The preliminary test

Obviously a very important test to learn.

For a series to be convergent a necessary but not sufficient condition is that the terms approach zero, $\lim_{n\to\infty} a_n = 0$.

1.5.1 Problems

Use preliminary test to decide if the series are divergent, or *require more testing* (one can never say they are convergent using the preliminary test).

Chapter 1 Problem 5.1

$$\frac{1}{2} - \frac{4}{5} + \frac{9}{10} - \frac{16}{17} + \frac{25}{26} - \frac{36}{37} + \dots$$

The general rule is helpful in applying the preliminary test. Here it is:

$$S = \sum_{n=1}^{\infty} -\frac{n^2}{n^2+1}(-1)^n$$

Here

$$\lim_{n\to\infty} a_n = \pm 1 \neq 0$$

so the series is divergent.

Chapter 1 Problem 5.2

$$\frac{\sqrt{2}}{1} + \frac{\sqrt{3}}{2} + \frac{\sqrt{4}}{3} + \frac{\sqrt{5}}{4} + \frac{\sqrt{6}}{5} + \dots$$

The general rule is helpful in applying the preliminary test. Here it is:

$$S = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}$$

Here

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n+1}}{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

so the series is not necessarily divergent. Further testing is required to decide if it is convergent.

Chapter 1 Problem 5.3

$$S = \sum_{n=1}^{\infty} \frac{n+3}{n^2+10\,n}$$

Here

$$\lim_{n\to\infty} a_n = \frac{n+3}{n^2+10\,n} = \lim_{n\to\infty} \frac{n}{n^2+10\,n} = \lim_{n\to\infty} \frac{1}{n+10} = \lim_{n\to\infty} \frac{1}{n} = 0$$

so the series is not necessarily divergent. Further testing is required to decide if it is convergent.

I'm skipping the even numbered problems because it's getting tedious.

Chapter 1 Problem 5.5

$$S = \sum_{n=1}^{\infty} \frac{n!}{n!+1}$$

Here

$$\lim_{n\to\infty} a_n = \frac{n!}{n!+1} = \lim_{n\to\infty} \frac{n!}{n!} = 1$$

so the series is divergent.

Chapter 1 Problem 5.7

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n\,n}{\sqrt{n^3+1}}$$

Here

$$\lim_{n\to\infty} a_n = \frac{(-1)^n\, n}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{(-1)^n\, 1}{\sqrt{n}} = 0$$

so the series is not necessarily divergent. Further testing is required.

Chapter 1 Problem 5.8 This one looked interesting:

$$S = \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

Here

$$\lim_{n\to\infty} a_n = \frac{\ln(n)}{n} = \lim_{n\to\infty} \ln(n^{\frac{1}{n}}) = 0$$

so the series is not necessarily divergent. Further testing is required. It helped me here to picture the graph of $\ln(x)$ vs x.

Chapter 1 Problem 5.11 Prove that the preliminary test reveals divergent series.

This is the most interesting problem.

Definition (4.6) requires that the partial sums of an infinite series, S_n, converge to the same limit, S, for the series to be convergent. We wish to prove this cannot be the case for an infinite series with nonzero limit for the terms a_n. By (4.6)

$$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

But

$$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} a_n$$

as well, which proves the preliminary test.

1.6 Tests for convergence of series of positive terms: Absolute convergence

"absolute convergence" means the sum of the absolute values converges.

A. The Comparison Test

1.6.1 Problems

Chapter 1 Problem 6.1 Show that $n! > 2^n$ for $n > 3$. $4! = 24 > 2^4 = 16$. Each subsequent term in 2^n is just twice the previous, while the subsequent terms of $n!$ are n times the previous.

Chapter 1 Problem 6.2 The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

has terms a_n always greater than or equal to the infinite series formed from

$$1 + (1/2) + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + (8 \text{ terms each of which is } 1/16) + \dots$$

The above diverges because it can be written as

$$1 + 1/2 + 1/2 + 1/2 \dots$$

Note that the $n = 2^m$, where $m \in \mathbb{N}$ terms of the above are equal to those of the harmonic series $1/n$. And the previous $n/2 - 1$ terms are always greater in the harmonic series,

$$\frac{1}{n - n/2} + \frac{1}{n - n/2 + 1} + \dots + \frac{1}{n - 1}$$

are all greater than

$$\frac{1}{n}$$

By the comparison test, the harmonic series diverges.

Chapter 1 Problem 6.3 This infinite series is convergent

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Hint: try grouping as in previous problem.

For $n > 6$, $1/n^2 > 1/2^n$, so we cannot compare to the geometric series. Stuck!

Chapter 1 Problem 6.4

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

This converges because $\frac{1}{n\,2^n}$.

Chapter 1 Problem 6.5a

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This diverges because $\frac{1}{\sqrt{n}} > \frac{1}{n}$.

Chapter 1 Problem 6.5a

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

This diverges because $\frac{1}{\ln(n)} > \frac{1}{n}$.

Chapter 1 Problem 6.6 There are 9 1-digit numbers, 90 2-digit numbers, 900 3-digit numbers. The pattern repeats because for n-digit numbers we have $(10^{(n-1)}), (10^{(n-1)} + 1), \ldots, (10^n - 1)$ for which there are $10^n - 10^{(n-1)}$. The harmonic series has terms always greater than or equal to series with

$$\left[\frac{1}{10} + \frac{1}{10} + (9\,\text{terms each of }\frac{1}{10})\right] + \left[90\,\text{terms each of }\frac{1}{100}\right] + \ldots = \frac{9}{10} + \frac{9}{10} + \ldots$$

That is, the harmonic series has each $n = 10^p, p \in Z$ equal to the n^{th} term in the above series. But all the corresponding terms in the harmonic series are all greater because they are $1/m > 1/n$ for $m < n$. The The latter series obviously diverges. By the comparison test, the harmonic series also diverges.

B. The Integral Test

Problems 6.7 through 6.14, use the integral test.

Chapter 1 Problem 6.7

$$\sum_{n=2}^{\infty} \frac{1}{n\ \ln(n)}$$

Compare to

$$\int^{\infty} \frac{1}{x\ \ln(x)}\, dx$$

Let $y = \ln(x)$, so $dy = 1/x\, dx$ and the above integral can be written:

$$\int^{\infty} \frac{1}{y}\, dy = \infty$$

So the series diverges.

Chapter 1 Problem 6.8

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4}$$

Compare to

$$\int^{\infty} \frac{x}{x^2+4}\, dx$$

Let $y = (x^2+4)/2$, so $dy = x\, dx$ and the above integral can be written:

$$\frac{1}{2}\int^{\infty} \frac{1}{y}\, dy = \infty$$

So the series diverges.

Chapter 1 Problem 6.9

$$\sum_{n=3}^{\infty} \frac{1}{n^2-4} = \sum_{n=3}^{\infty} \frac{1}{(n-2)(n+2)} < \sum_{n=3}^{\infty} \frac{1}{(n-2)^2}$$

Compare to

$$\int^{\infty} \frac{1}{(x-2)^2}\, dx$$

Let $y = (x - 2)$, so $dy = dx$ and the above integral can be written:

$$\int^{\infty} \frac{1}{y^2}\, dy = -\left[\frac{1}{y}\right]^{\infty} = 0$$

So the series converges.

Chapter 1 Problem 6.10

$$\sum_{n=1}^{\infty} \frac{\exp(n)}{\exp(2n)+9}$$

Compare to

$$\int^{\infty} \frac{\exp(x)}{\exp(2x)+9} dx < \int^{\infty} \frac{\exp(x)}{\exp(2x)} dx$$

Let $y = \exp(x)$, so $dy = \exp(x)\, dx$ and the above integral can be written:

$$\int^{\infty} \frac{1}{y^2}\, dy = -\left[\frac{1}{y}\right]^{\infty} = 0$$

So the series converges.

Skipping some...

Chapter 1 Problem 6.16

The mistake of including a lower limit on the integral test led to crazy results. It can be understood by recognizing that the integral test is really a form of comparison test. But the comparison depicted in Fig. 6.2, between

$$\sum_{n=1}^{\infty} \frac{1}{n} < \int_{n=0}^{\infty} \frac{1}{x}\, dx$$

can only be used to show that a series converges. In the case that the reference series (or in this case reference integral) diverges, we can say nothing about the series under consideration. This was the essence of the mistake the student made.

Chapter 1 Problem 6.17

$$\sum_{n=0}^{\infty} \exp(-n^2) < \sum_{n=0}^{\infty} \exp(-n)$$

She hints that we can compare to the integral $\exp(-x)$ since this is easier.

$$\int^{\infty} \exp(-x)dx = 0$$

So the series converges. The point is that if your careful about the function you chose to compare to, you might save some work in doing the integral.

C. The Ratio Test

Problems 6.18 through 6.30 involve the ratio test

Chapter 1 Problem 6.18

$$\sum_{1}^{\infty} \frac{2^n}{n^2}$$

In the numerator, the power gets large, while in the denominator, the base gets large. We'll see that it's the power that matters more than the base. The ratio

$$\rho_n = \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{n+1} \frac{n}{2^n} = \frac{2n}{n+1}$$

The limit is obvious,

$$\rho \equiv \lim_{n \to \infty} \rho_n = 2 > 1$$

so the series must diverge.

Chapter 1 Problem 6.19

$$\sum_{0}^{\infty} \frac{3^n}{2^{2n}}$$

Note there are two ways to approach this. One could simply write the series as a geometric series,

$$\sum_{0}^{\infty} \frac{3^n}{2^{2n}} = \sum_{0}^{\infty} \frac{3^n}{(2^2)^n} = \sum_{0}^{\infty} \left(\frac{3}{4}\right)^n$$

which is clearly convergent. Using the ratio test to the series as written leads to

$$\rho_n = 3/4$$

so the ratio test also implies this series is convergent.

Chapter 1 Problem 6.21

$$\sum_0^\infty \frac{5^n\,(n!)^2}{(2n)!}$$

Here the factorial in the denominator will turn out to get large the fastest, and overwhelm even the geometric term 5^n and the squared factorial.

$$\rho_n = \frac{5^{(n+1)}(n+1!)^2}{(2(n+1))!}\frac{(2n)!}{5^n\,(n!)^2}$$

The only trick is to write the factorial terms as $(n+1)! = (n+1)n!$, so that the factorial in the denominator can be canceled. After some algebra,

$$\rho_n = \frac{5(n+1)^2}{(2n+1)(2n+2)}$$

So the limit

$$\rho = 5/4$$

and the series diverges.

Chapter 1 Problem 6.23

$$\sum_1^\infty \frac{n!}{100^n}$$

Here,

$$\rho_n = \frac{(n+1)n!}{100\cdot 100^n}\frac{100^n}{n!} = \frac{n+1}{100}$$

So

$$\rho = \infty$$

and the series diverges. Factorial gets large really fast, even faster than the exponential function with a "big" base.

Chapter 1 Problem 6.24

$$\sum_{0}^{\infty} \frac{3^{2n}}{2^{3n}}$$

Here,

$$\rho_n = \frac{3^{2(n+1)}}{2^{3(n+1)}} \frac{2^{3n}}{3^{2n}} = \frac{3^2}{2^3} = \frac{9}{8}$$

So

$$\rho = \frac{9}{8}$$

and the series diverges.

Chapter 1 Problem 6.25

$$\sum_{0}^{\infty} \frac{\exp(n)}{\sqrt{n!}}$$

Here,

$$\rho_n = \frac{\exp(n+1)}{\sqrt{(n+1)!}} \frac{\sqrt{n!}}{\exp(n)} = \frac{\mathrm{e}}{\sqrt{n+1}}$$

So

$$\rho = 0.$$

and the series converges.

Chapter 1 Problem 6.27

$$\sum_{0}^{\infty} \frac{100^n}{n^{200}}$$

Here,

$$\rho_n = \frac{100^{(n+1)}}{(n+1)^{200}} \frac{n^{200}}{100^n} = 100 \frac{n^{200}}{(n+1)^{200}}$$

So

$$\rho = 100.$$

and the series diverges. The exponential term overpowers the power term!

Chapter 1 Problem 6.29

$$\sum_0^\infty \frac{\sqrt{(2n)!}}{n!}$$

Here,

$$\rho_n = \frac{\sqrt{(2(n+1))!}}{(n+1)!}\frac{n!}{\sqrt{(2n)!}} = \frac{1}{n+1}\sqrt{(2n+1)(2n+2)} > 1$$

So

$$\rho = 4.$$

and the series diverges. The exponential term overpowers the power term!

Chapter 1 Problem 6.30 Prove the ratio test. This is perhaps the most important problem for understanding what we're doing. However, the previous problems are important for gaining skill. She again points us to the answer with a helpful hint. For a given infinite series $\sum_{n=0}^\infty a_n$, we consider the limit to exist:

$$\lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$$

where $\rho < 1$. That is, we imagine an infinite series that passes the ratio test. Consider some σ such that $\rho < \sigma < 1$. Then there is a value of N such that $\rho_n < \sigma$ for all $n \geq N$. We might have to go to very large N, but the above limit ensures that N exists. Now we construct a geometric series

$$\sum_{n=0}^\infty a\sigma^n$$

with term $a\sigma^N = a_N$. The two series cross at $n = N$, but for $n > N$ we always have $a_n < a\sigma^n$ by the definition of N. This seems obvious and can be proved by induction. For $n = N + 1$ we have

$$a_{N+1} = \rho_n \, a_N < \sigma \, a_N = a\sigma^{N+1}$$

And furthermore if for some arbitrary $n > N$ we have that $a_n < a\sigma^n$, then it's also clear that $a_{n+1} < a\sigma^{n+1}$ by a similar argument. This completes the proof by induction that $a_n < a\sigma^n$ for all $n > N$. The geometric series is convergent and by the comparison test we have that the given series is convergent. This proves the ratio test.

D. A Special Comparison Test

For an infinite series with positive terms, $a_n \geq 0$, to prove the series converges we only need to prove that the ratio of terms a_n/b_n tends to a finite limit, where $b_n \geq 0$ is any positive, convergent infinite series. The trick is in finding the appropriate comparison series b_n.

Chapter 2

Complex Numbers

Chapter 3

Linear equations: vectors, matrices, and determinants

Chapter 4

Partial differentiation

Chapter 5

Multiple integrals: applications of integration

Chapter 6

Vector Analysis

Chapter 7

Fourier Series

Chapter 8

Ordinary differential equations

Chapter 9

Calculus of Variations

9.1 Introduction

How do you find stationary values of integral quantities?

9.1.1 Problems

Chapter 9 Problem 1.1 Derive the optical law of reflection.

Light leaves point $A = (x_1, y_1)$, reflects off point $P(x, 0)$, and reaches point $B(x_2, y_2)$. Let $D = APB$ be the total distance traveled. Then the time of travel, $t = D/c$. Find point P using principle that t is stationary. That is,

$$\frac{dt}{dx} = \frac{1}{c}\frac{dD}{dx} = 0$$

This requires

$$c\frac{dt}{dx} = 0 = \frac{dD}{dx} = \frac{d}{dx}\left(\sqrt{(x-x_1)^2+y^2} + \sqrt{(x_2-x)^2+y^2}\right) = \sin(\theta_1) + \sin(\theta_2),$$

where $\sin(\theta_1) = (x - x_1)/AP$, and $\sin(\theta_2) = (x_2 - x)/PB$. So $\theta_1 = \theta_2$.

Chapter 9 Problem 1.2 Derive Snell's Law of refraction.

Light passes from medium with index of refraction n_1 into medium with n_2. Use similar geometry to Problem 1 above. Now, say, $y_1 > 0$ and $y_2 < 0$, and the line $y = 0$ divides the two media. Again we can find point P by assuming that t is stationary. But now,

$$c\,t = n_1 AP + n_2 PB.$$

So now, this requires

$$\frac{dt}{dx} = \frac{1}{c}\frac{d}{dx}\left(n_1\sqrt{(x-x_1)^2+y^2} + n_2\sqrt{(x_2-x)^2+y^2}\right) = n_1\sin(\theta_1) + n_2\sin(\theta_2) = 0,$$

which is Snell's Law of refraction.

It's instructive to attempt to derive the derivative of distance from the trignometric relationship

$$AP = \frac{x}{\sin(\theta_1)}$$

If you naively differentiate this wrt x, you might obtain,

$$\frac{d}{dx}AP = \frac{d}{dx}\left(\frac{x}{\sin(\theta_1)}\right) = \frac{1}{\sin(\theta_1)} - \frac{x}{\sin^2(\theta_1)}\frac{d}{dx}\sin(\theta_1).$$

This looks so complicated, I thought at first that it was wrong. Darran Furnival found that it is actually correct, along obviously not in a nice form!

Chapter 9 Problem 1.3 Show the actual path is not necessarily a minimum. Have light from source A reflecting off a curved mirror, and received at point B.

(a) Show that if A and B are on the focii of an ellipse, then the varied paths all have the same length.

This is a defining property of an ellipse, so in one sense there is nothing to do. However, I took this as a request to show that this defining property of an ellipse gives you the canonical equation for an ellipse. I actually got stuck and had to find the solution on Wikipedia.

Draw an ellipse with center at the origin, and focci at $x = \pm c$. Let the length of the major axis (minor axis) be $a(b)$. Distance from $(-c, 0)$ to a point (x, y) is d_2, and from the point (x, y) to $(c, 0)$ is d_1. Then $d = d_1 + d_2$ should be a constant – it's the defining property of an ellipse. Try to derive the canonical form.

$$d_1 = \sqrt{(c - x)^2 + y^2} \tag{9.1}$$

$$d_2 = \sqrt{(c + x)^2 + y^2} \tag{9.2}$$

Let

$$d_1 + d_2 = 2\,a$$

where a is a constant (it's the major axis length). Thus

$$d_1^2 = 4a^2 - 4\,a\,d_2 + d_2^2$$

Substitute for d_1^2, d_2^2, and d_2. Note the x^2, y^2, and c^2 terms appear on both sides of the $=$ sign, and therefore cancel, leaving only

$$-2\,c\,x = 4\,a^2 - 4\,a\,\sqrt{(c + x)^2 + y^2} + 2\,x\,c$$

Rearranging gives

$$\frac{c\,x}{a} + a = \sqrt{(c + x)^2 + y^2}$$

Square both sides, and note that the $2\,c\,x$ terms cancel, Rearranging gives the desired canonical form of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2 - c^2$.

(b) and (c) where visually obvious, so I'm not sure what she expected us to do.

9.2 Euler Equation

The standard problem involves always solving the same ordinary differential equation. This is called the Euler Equation.

I found that if you complicate the problem by introducing second derivatives, then the Euler Equation is not the relevant ODE.

9.2.1 Problems: Write and solve the Euler eqn. to make the definite integrals stationary.

Chapter 9 Problem 2.1

$$\int_{x_1}^{x_2} \sqrt{x}\,\sqrt{1+y'^2}\,dx$$

Here,

$$F(x,y,y') = \sqrt{x}\sqrt{1+y'^2}$$

Because

$$\frac{\partial F}{\partial y} = 0$$

the Euler equation can immediately be integrated wrt x to give,

$$\frac{\partial F}{\partial y'} = c_1$$

where c_1 is a constant. So then

$$\frac{y'}{\sqrt{1+y'^2}} = c_1/\sqrt{x}$$

I've found the following manipulation comes up a lot:

$$a = \frac{b}{\sqrt{1+b^2}}$$

which rearranges to

$$b = \pm\frac{a}{\sqrt{1-a^2}}$$

We can immediately isolate y':

$$y' = \frac{c_1/\sqrt{x}}{\sqrt{1-c_1^2/x}} = \frac{c_1}{\sqrt{x-c_1^2}}$$

where the $\pm$ has been absorbed in c_1. This can be integrated by inspection,

$$y(x) = 2\,c_1\sqrt{x-c_1^2} + c_2.$$

The constants c_1 and c_2 can be chosen so that $y(x_1)$ and $y(x_2)$ meet whatever BCs are given there.

Chapter 9 Problem 2.2

$$\int_{x_1}^{x_2} \frac{ds}{x} = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x} dx$$

$$F(x, y, y') = \frac{\sqrt{1+y'^2}}{x} \tag{9.3}$$

$$\tag{9.4}$$

Because

$$\frac{\partial F}{\partial y} = 0$$

we can immediately write,

$$\frac{\partial F}{\partial y'} = c_1$$

where c_1 is a constant. So then

$$\frac{y'}{x\sqrt{1+y'^2}} = c_1$$

Rearranging

$$\frac{c_1\, x'}{\sqrt{1-c_1^2\, x'^2}} = y'$$

Let $c_1 x = \sin(\theta)$ and this can be integrated easily:

$$y(x) = \pm\frac{1}{c_1}\sqrt{1-c_1^2\, x^2} + c_2$$

Chapter 9 Problem 2.3

$$\int_{x_1}^{x_2} x\sqrt{1-y'^2}dx$$

Here

$$F(x,y,y') = x\sqrt{1-y'^2}$$

and again,

$$\frac{\partial F}{\partial y} = 0$$

we can immediately write,

$$\frac{\partial F}{\partial y'} = c_1$$

where c_1 is a constant. Rearranging gives,

$$\frac{-x\, y'}{\sqrt{1-y'^2}} = c_1$$

Square both sides, and isolate y':

$$y' = \pm\frac{c_1}{\sqrt{x^2+c_1^2}}$$

Note that the $\pm$ can be absorbed into the arbitrary constant c_1. Now we have to solve an integral:

$$y(x) = c_2 + \int \frac{c_1}{\sqrt{x^2 + c_1^2}}\, dx = c_1 \ln\left(x + \sqrt{x^2 + c_1^2}\right) + c_2$$

I got this from Maple, but it should be in tables or other symbolic mathematics programs. It will come up a lot, and is probably worth remembering. Differentiating confirms this is the solution to the integral.

Chapter 9 Problem 2.4

$$\int_{x_1}^{x_2} x\sqrt{1 + y'^2}dx$$

Here

$$F(x, y, y') = x\sqrt{1 + y'^2}$$

and again,

$$\frac{\partial F}{\partial y} = 0$$

we can immediately write,

$$\frac{\partial F}{\partial y'} = c_1$$

where c_1 is a constant. Rearranging gives,

$$y = \int \frac{c_1}{\sqrt{x^2 - c_1^2}}dx + c_2$$

Stuck on this integral!

Update: p. 391, she solves a problem involving this integral. The Answer is:

$$y(x) = c_1 \cosh^{-1}(x/c_1) + c_2.$$

I'm now stuck on how to prove this is the antiderivative because I don't know how to differentiate $\cosh^{-1}$. I tried implicit differentiation, but got stuck.

Update: The above integral is similar to that in problem 2.3. That is, one can also write:

$$y(x) = c_1 \ln\left(x + \sqrt{x^2 - c_1^2}\right) + c_2$$

And, as in problem 2.3, it's trivial to show this is the antiderivative. I guess she likes the hyperbolic trignometric function representation for some reason, though I find it more straightforward to use the elementary function.

Chapter 9 Problem 2.5. Here

$$F(x, y, y') = y'^2 + y^2$$

Now we must write the full Euler equation,

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 = 2y'' - 2y$$

By inspection

$$y(x) = c_1 \exp(x) + c_2 \exp(-x).$$

Chapter 9 Problem 2.6. Here

$$F(x, y, y') = y'^2 + y^{1/2}$$

Again we must write the full Euler equation,

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 = 2y'' - \frac{1}{2\sqrt{y}}$$

This looks tough – a 2nd order, nonlinear ODE.

Update: After reading Section 9.3 it became clear that a change of variables simplifies the problem.

Note that $y' = 1/x'$, (though she never talks about when $x' = 0$ – what happens then?) and so

$$\int_{x_1}^{x_2} y'^2 + y^{1/2}\,dx = \int_{x_1}^{x_2} \frac{1}{x'}\,y' + x'\,y'\,y^{1/2}\,dx \tag{9.5}$$

$$= \int_{y_1}^{y_2} \frac{1}{x'} + x'\,y^{1/2}\,dy \tag{9.6}$$

where clearly $y_1 = y(x_1)$ and $y_2 = y(x_2)$. Now we can immediately write

$$\frac{\partial}{\partial x'}\left(\frac{1}{x'} + x'\,y^{1/2}\right) = c_1 = \frac{-1}{x'^2} + y^{1/2}$$

Rearranging gives,

$$dx = \frac{1}{\sqrt{\sqrt{y} - c_1}}\,dy$$

Now I'm stuck on this integral!

Important lesson: Apparently sometimes the problems are setup to motivate later sections. If something looks hard, don't despair because later on we might learn something that will make it easier.

Update: Integral has a simple but messy form when $c_1 < 0$, found in Maple.

Chapter 9 Problem 2.7. Here

$$F(x, y, y') = \exp(x)\,\sqrt{1 + y'^2}$$

There's no y dependence, so we can immediately write

$$\frac{\partial F}{\partial y'} = c_1 = \exp(x)\frac{y'}{\sqrt{1 + y'^2}}$$

Using the handy relation (see problem 1),

$$y' = \frac{c_1 \exp(-x)}{\sqrt{1 - (c_1\,\exp(-x))^2}} = \frac{c_1}{\sqrt{\exp(2x) - c_1^2}}$$

Boas suggests using $u = \exp(x)$, so $dx = du/u$, which leads to an integral of the form

$$dy = \frac{c_1}{u\sqrt{u^2 - c_1^2}} du$$

which unfortunately is not more obvious to me that before I took her suggestion!

Update: Being a little more persistent, I find

$$dy = \frac{c_1 \exp(-x)}{\sqrt{1 - (c_1 \exp(-x))^2}} dx \tag{9.7}$$

$$= \frac{c_1 1/u^2}{\sqrt{1 - (c_1^2 (1/u^2)}} du \tag{9.8}$$

$$= \frac{c_1}{\sqrt{u^2 - c_1^2}} du \tag{9.9}$$

We solved this integral in problem 4.

$$y(u) = c_1 \cosh^{-1}(u/c_1) + c_2$$

so

$$y(x) = c_1 \cosh^{-1}\left(\frac{\exp(x)}{c_1}\right) + c_2.$$

This probably simplifies, but not immediately clear how.

9.3 Using the Euler Equation

If you change variables, sometimes you can obtain an $F(x, y, y') = F(x, y')$, so that the Euler Eqn simplifies to

$$\frac{\partial F}{\partial y'} = c_1.$$

This is called a "first integral of Euler's Eqn.".

Given two points in the $x - y$ plane, the curve joining them that forms the surface with minimum area when revolved about the x-axis is called a "catenary".

9.3.1 Problems

Find first integral of Euler Eqn. by changing variables, 1 to 4.

1.

$$\int_{x_1}^{x_2} y^{3/2}\, ds$$

$$\int_{x_1}^{x_2} y^{3/2}\, ds = \int_{x_1}^{x_2} y^{3/2}\, \sqrt{dx^2 + dy^2} \tag{9.10}$$

$$= \int_{y_1}^{y_2} y^{3/2}\, \sqrt{x'^2 + 1}\, dy \tag{9.11}$$

The Euler Eqn simplifies to

$$\frac{\partial y^{3/2}\, \sqrt{x'^2 + 1}}{\partial x'} = c_1.$$

Taking the derivative,

$$y^{3/2} \frac{x'}{\sqrt{x'^2 + 1}} = c_1.$$

which reduces to

$$dx = \frac{c_1}{\sqrt{y^3 - c_1^2}}\, dy$$

2.

$$\int_{x_1}^{x_2} y^{-2} \sqrt{1 + y'^2}\, dx$$

$$\int_{x_1}^{x_2} y^{-2} \sqrt{1 + y'^2}\, dx = \int_{x_1}^{x_2} y^{-2}\, \sqrt{dx^2 + dy^2} \tag{9.12}$$

$$= \int_{x_1}^{x_2} y^{-2}\, ds \tag{9.13}$$

$$= \int_{x_1}^{x_2} y^{-2}\, \sqrt{1 + x'^2}\, dy \tag{9.14}$$

So $F(y,x,x') = y^{-2}\sqrt{1+x'^2}$ and

$$\frac{\partial F}{\partial x} = 0$$

Thus we can immediately write the first integral of Euler's Eqn.

$$\frac{\partial F}{\partial x'} = y^{-2}\,\frac{x'}{\sqrt{1+x'^2}} = c_1$$

This rearranges to

$$x' = \frac{y^2\,c_1}{\sqrt{1-y^4\,c_1^2}}$$

which gives a tricky integral

$$x = \int \frac{y^2\,c_1}{\sqrt{1-y^4\,c_1^2}}\,dy + c_2$$

3.

$$\int_{y_1}^{y_2} \frac{x'^2}{\sqrt{x+x'^2}}\,dy$$

Use one of the x' in the numerator to cancel the dy and give a dx. Write the other as $y' = 1/x'$.

$$\int_{y_1}^{y_2} \frac{x'^2}{\sqrt{x+x'^2}}\,dy = \int_{x_1}^{x_2} \frac{1}{\sqrt{1+x\,y'^2}}\,dx$$

$$\frac{\partial F}{\partial y} = 0$$

so we can immediately write the first integral of Euler's Eqn.

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{1+x\,y'^2}} = c_1$$

and so,

$$\frac{\partial F}{\partial y'} = \frac{\partial}{\partial y'}\frac{1}{\sqrt{1+x\,y'^2}} = c_1$$

which simplifies to

$$\frac{-x\,y'}{(1+x\,y'^2)^{3/2}} = c_1$$

Again a nasty integral, but we're not actually asked to solve the integral.

4.

$$\int_{x_1}^{x_2} y\,\sqrt{y^2 + y'^2}\,dx = \int_{x_1}^{x_2} y\,\sqrt{x'^2\,y^2 + 1}\,y'\,dx \tag{9.15}$$

$$= \int_{y_1}^{y_2} y\,\sqrt{x'^2\,y^2 + 1}\,dy \tag{9.16}$$

Euler's first integral is

$$\frac{\partial F}{\partial x'} = \frac{y^3\,x'}{\sqrt{1 + x'^2\,y^2}} = c_1$$

Find $y(x)$ that makes the following integrals stationary. Use change of variables if necessary.

5.

$$\int_{x_1}^{x_2} \sqrt{1 + y^2\,y'^2}\,dx$$

Changing variables simplified the above to

$$\frac{\partial}{\partial x'}\sqrt{x'^2 + y^2} = c_1$$

So,

$$\frac{x'}{\sqrt{x'^2 + y^2}} = c_1$$

Rearranging

$$dx = c_2\,y^{1/2}\,dy$$

$$x(y) = c_3\,y^{3/2} + c_4$$

6.

$$\int_{x_1}^{x_2} \frac{y\,y'^2}{1 + y\,y'}\,dx$$

Changing variables simplified the above to

$$\int_{y_1}^{y_2} \frac{y}{x' + y}\, dy$$

Because

$$\frac{\partial}{\partial x} \frac{y}{x' + y} = 0$$

we can immediately write the first integral of the Euler Eqn.

$$\frac{\partial}{\partial x'} \left(\frac{y}{x' + y} \right) = c_1 = \frac{-y}{(x' + y)^2}$$

This rearranges to (after redefining the constant to c_2)

$$dx = (c_2 \sqrt{y} - y)\, dy$$

so clearly

$$x(y) = c_3\, y^{3/2} - \frac{y^2}{2} + c_4.$$

7.

$$\int_{x_1}^{x_2} y'^2 + y^2\, dx$$

Changing variables simplified the above to

$$\int_{y_1}^{y_2} \frac{1}{x'} + x'\, y^2\, dy$$

Because

$$\frac{\partial}{\partial x} \left(\frac{1}{x'} + x'\, y^2 \right) = 0$$

we can immediately write the first integral of the Euler Eqn.

$$\frac{\partial}{\partial x'} \left(\frac{1}{x'} + x'\, y^2 \right) = c_1 = \frac{-1}{x'^2} + y^2$$

which rearranges to

$$x' = \pm \frac{1}{\sqrt{y^2 - c_1}}$$

This is similar to the integral on p. 391 discussed above.

$$x(y) = \cosh^{-1} \frac{y}{c_1} + c_2$$

8.

$$\int_{\theta_1}^{\theta_2} \sqrt{r'^2 + r^2}\, d\theta = \int_{\theta_1}^{\theta_2} r' \sqrt{1 + r^2/r'^2} d\theta \tag{9.17}$$

$$= \int_{r_1}^{r_2} \sqrt{1 + r^2\theta'^2}\, dr \tag{9.18}$$

We can immediately write the first integral of the Euler Eqn.

$$\frac{\partial}{\partial \theta'} \left(\sqrt{1 + r^2\theta'^2} \right) = c_1 = \frac{r^2\theta'}{\sqrt{1 + \theta'^2 r^2}}$$

which rearranges to

$$\theta' = \pm \frac{1}{r\sqrt{1 - \frac{r^2}{c_1^2}}}$$

$$\theta(r) = \int \pm \frac{1}{r\sqrt{1 - \frac{r^2}{c_1^2}}} dr + c_2$$

Stuck on this integral.

9.

$$\int_{\phi_1}^{\phi_2} \sqrt{\theta'^2 + \sin^2\theta}\, d\phi = \int_{\phi_1}^{\phi_2} \theta' \sqrt{1 + \sin^2(\theta)/\theta'^2}\, d\phi \tag{9.19}$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{1 + \phi'^2 \sin^2\theta}\, d\theta \tag{9.20}$$

We can immediately write the first integral of the Euler Eqn.

$$\frac{\partial}{\partial \phi'} \left(\sqrt{1 + \phi'^2 \sin^2\theta} \right) = c_1 = \frac{\phi' \sin^2\theta}{\sqrt{1 + \phi'^2 \sin^2\theta}}$$

which rearranges to

$$\phi' = \pm \frac{c_1^2}{\sin\theta\sqrt{\sin^2\theta - c_1^2}}$$

$$\phi(\theta) = \pm \int \frac{c_1^2}{\sin\theta\sqrt{\sin^2\theta - c_1^2}} d\theta + c_2$$

Stuck on this integral.

10.

$$\int_{t_1}^{t_2} \frac{1}{s}\sqrt{s'^2 + s^2}\, dt = \int_{t_1}^{t_2} \frac{s'}{s}\sqrt{1 + s^2/s'^2}\, dt \tag{9.21}$$

$$= \int_{s_1}^{s_2} \frac{1}{s}\sqrt{1 + t'^2\, s^2}\, ds \tag{9.22}$$

We can immediately write the first integral of the Euler Eqn.

$$\frac{\partial}{\partial t'}\left(\frac{1}{s}\sqrt{1 + t'^2\, s^2}\right) = c_1 = \frac{t'\, s}{\sqrt{1 + t'^2\, s^2}}$$

which rearranges to

$$t' = \pm\frac{c_2}{s}$$

Finally, an easy integral:

$$t(s) = \pm c_2 \ln s + c_3.$$

Use Fermat's principle (see p. 385) to find the path followed by light through media with given index of refraction. Problems 11 to 14.

Note that $n \geq 1$ by definition, since light cannot travel faster than c. So there are implicit restrictions on domain.

11. $n = x + 1$. (Valid only for $x > 0$.)

Recall that $v = c/n$, where c is the speed of light in a vacuum, $c = 3.0 \times 10^8$m/s. Therefore

$$dt = ds/v = \frac{n(x)}{c}\, ds$$

It's best to write this with x as the independent variable so that the integrand is independent of the dependent variable y:

$$c\int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} (x+1)\sqrt{1+y'^2}\,dx$$

Then we can immediately write a first integral of the Euler Eqn.

$$\frac{\partial}{\partial y'}\left((x+1)\sqrt{1+y'^2}\right) = c_1 = \frac{(x+1)\,y'}{\sqrt{1+y'^2}}$$

which rearranges to

$$y' = \pm\frac{c_1}{\sqrt{(x+1)^2 - c_1^2}}$$

So

$$y(x) = \pm c_1\,\cosh^{-1}\left(\frac{x+1}{c_1}\right) + c_2$$

Question: What resolves the ambiguity with the sign?

12. $n = 1/y$. (Note, valid only when $y \leq 1$.) It's best to write this with y as the independent variable so that the integrand is independent of the dependent variable x:

$$c\int_{t_1}^{t_2} dt = \int_{y_1}^{y_2} \frac{1}{y}\sqrt{1+x'^2}\,dy$$

Then we can immediately write a first integral of the Euler Eqn.

$$\frac{\partial}{\partial x'}\left(\frac{1}{y}\sqrt{1+x'^2}\right) = c_1 = \frac{1}{y}\frac{x'}{\sqrt{1+x'^2}}$$

which rearranges to

$$x' = \pm\frac{c_1\,y}{\sqrt{1-y^2\,c_1^2}}$$

$$dx = \pm\frac{c_1\,y}{\sqrt{1-y^2\,c_1^2}}\,dy$$

By inspection

$$x(y) = \pm\frac{1}{c_1}\sqrt{1 - y^2\, c_1^2} + c_2$$

which is a circle

$$(x - c_2)^2 + y^2 = \frac{1}{c_1^2}.$$

9-3. 13. $n = \sqrt{y}$ The velocity of light is $v = c/n(y)$. To make the integral of the time stationary, the Euler eqn must be satisfied:

$$\frac{d}{dy}\left(\frac{\partial F(y, x, x')}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$$

9.4 The Brachistochrone problem: cycloids

9.5 Several dependent variables: Lagrange's Equations

9.5.1 Problems

9.5 1.

Derive the Euler eqns for the case where $F(x, y, z, y', z')$, and we want to make

$$I = \int_{x_1}^{x_2} F dx$$

stationary.

This is very straightforward, and she walks you through the solution with hints.

The subtleties arise from a) knowing how to form F. Does it depend upon y''? or some higher derivative? If so, then the derivation would change, b) Does the solution have infinite derivative at the boundaries? $y' \to \infty$? This would mess up the integration by parts because we could no longer through out the boundary terms.

9.5 Problem 2. For potential $V(r, \theta, z)$ find equations of motion in cylindrical coordinates.

This is a straightforward generalization of the examples given in section 9.5. We need to construct the Lagrangian as a function of t, and the cylindrical coordinate variables parameterized by t: The Lagrangian is

$$L = L(r, \theta, z, \dot{r}, \dot{\theta}, \dot{z}) = \frac{1}{2}mv^2 - V$$

So we need to find the velocity v as a function of these variables. We start, as she hints, with an infinitesimal displacement ds,

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2.$$

From previous problems of this chapter we are used to parameterizing ds in terms of one of the position variables, like r, but that would be a wrong turn here. That would introduce variables $d\theta/dr$ etc. and remember we want the $L = L(r, \theta, z, \dot{r}, \dot{\theta}, \dot{z})$. So instead we divide by dt^2 to obtain (in the limit of $dt \to 0$, which apparently is implicit in the notation dt)

$$\frac{ds^2}{dt^2} = \dot{r}^2 + r^2\, \dot{\theta}^2 + \dot{z}^2 = v^2.$$

Aside: One of my undergraduate mathematics professors at Queen's University warned us against doing the step we just did. "You can't just divide by a differential" I believe he said. But of course you can. You just have to be careful to think about what you're doing. You're considering the ratio of an infinitesimal displacement ds in cylindrical coordinates in infinitesimal time dt. You can of course be more thorough and consider the ratio of a finite displacement Δs in finite time Δt, and then take the limit of $\Delta t \to 0$. But the result is obvious!

Back to the problem: We now have the Lagrangian:

$$L = \frac{1}{2}\, m\, \dot{r}^2 + r^2\, \dot{\theta}^2 + \dot{z}^2 - V(r,\theta,z)$$

We now simply find the Euler equation for each component. For the radial component:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

So the radial acceleration is balanced by a potential gradient and the artificial centrifugal force

$$m\,\ddot{r} = -\frac{\partial V}{\partial r} + m\, r\, \dot{\theta}^2$$

For the angular component:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

So the angular acceleration is balanced by a potential gradient and the artificial Coriolis force

$$m\, r\, \ddot{\theta} = -\frac{\partial V}{\partial \theta} - 2m\, \dot{r}\, \dot{\theta}$$

The vertical component is unchanged from Cartesian coordinates, as in Example 1 of Section 9.5.

$$m\, \ddot{z} = -\frac{\partial V}{\partial z}$$

9.5 Problem 3. For potential $V(r,\theta,z)$ find equations of motion in cylindrical coordinates.

This is a straightforward extension of Problem 2.

We start again with an infinitesimal displacement ds,

$$v^2 = \frac{ds^2}{dt^2} = \frac{dr^2}{dt^2} + r^2\, \frac{d\theta^2}{dt^2} + r^2\, \sin^2(\theta)\, \frac{d\phi^2}{dt^2} \tag{9.23}$$

$$= \dot{r}^2 + r^2\, \dot{\theta}^2 + r^2\, \sin^2(\theta)\, \dot{\phi}^2 \tag{9.24}$$

which we'll use in the Lagrangian:

$$L(t,r,\theta,\phi) = \frac{1}{2}m\,\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\sin^2(\theta)\,\dot{\phi}^2\right) - V(r,\theta,\phi)$$

r direction:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

leads to

$$m\,\left(\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right) = -\frac{\partial V}{\partial r}$$

The extra acceleration term in the centrepetal acceleration, which has two contributions – one for zonal and one for meridional motion.

In the meridional direction:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

The left term leads to:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mr^2\dot{\theta}\right) \tag{9.25}$$

$$= m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) \tag{9.26}$$

The right term leads to:

$$-\frac{\partial L}{\partial \theta} = -mr^2\,\sin(\theta)\,\cos(\theta)\dot{\phi}^2 + \frac{\partial V}{\partial \theta}$$

Putting these two terms together, dividing by r,

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\,\sin(\theta)\,\cos(\theta)\dot{\phi}^2) = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

The extra acceleration terms here must be Coriolis terms, though it's not immediately obvious how they come about.

The zonal direction:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

There is no explicit ϕ dependence, so the equation is more straightforward, although the time derivatives leads to 3 terms

$$m(r^2\,\sin^2\theta\ddot{\phi} + 2r\dot{r}\,\sin^2\theta\dot{\phi} + 2\,r^2\,\sin\theta\dot{\theta}\dot{\phi}) + \frac{\partial V}{\partial \phi}$$

Dividing through by r,

$$m(r\,\sin^2\theta\ddot{\phi} + 2\dot{r}\,\sin^2\theta\dot{\phi} + 2\,r\,\sin\theta\dot{\theta}\dot{\phi}) = -\frac{1}{r}\frac{\partial V}{\partial \phi}$$

Chapter 9 Problem 5.4

Find equations of motion for a simple pendulum, see Chapter 8 problem 5.34.

Mass m, length l, angle to vertical θ. One must make an assumption that the string holding the mass is alway taught. Then the potential energy is simply

$$V(\theta) = mgl(1 - \cos(\theta))$$

The kinetic energy is simply

$$\mathrm{KE} = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

So the Lagrangian is

$$L = \mathrm{KE} - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos(\theta))$$

Note there is only one independent variable, θ. The governing equation follows from the single Euler equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{9.27}$$

$$m\,l^2\ddot{\theta} - m\,g\,l\,\sin(\theta) = 0 \tag{9.28}$$

$$\ddot{\theta} = \frac{g}{l}\,\sin(\theta) \tag{9.29}$$

For small θ this is simple harmonic motion, *i.e.* sinusoidal motion with period $2\pi\sqrt{l/g}$.

Chapter 9 Problem 5.5

Find equations of motion for motion along the $x-$axis with potential $V = 1/2kx^2$. This is simple harmonic motion.

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

So the Lagrangian is

$$L = \text{KE} - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\,m\,k\,x^2$$

Euler's equation immediately gives

$$m\ddot{x} - k\,x = 0$$

This is simple harmonic motion, *i.e.* sinusoidal motion with period $2\pi\sqrt{m/k}$.

Chapter 9 Problem 5.6

Motion on the surface of sphere of radius a, called a "spherical pendulum". Using spherical coordinates with $r = a$,

$$\frac{ds^2}{dt^2} = a^2\frac{d\theta^2}{dt^2} + a^2\sin^2(\theta)\frac{d\phi^2}{dt^2}$$

The potential energy is always constant and therefore does not affect the equations of motion:

$$L = \frac{1}{2}\,m\,\left(a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2\right) - V$$

Now we have two Euler equations. The one for θ gives,

$$\ddot{\theta} = 0$$

but the one for ϕ is more complicated (because the meridians converge on the sphere),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \tag{9.30}$$

$$\frac{d}{dt}\left(m\,a^2\,\sin^2(\theta)\,\dot{\phi}\right) = 0 \tag{9.31}$$

$$m\,a^2\,\left(2\,\sin(\theta)\cos(\theta)\dot{\theta}\,\dot{\phi} + \sin^2(\theta)\,\ddot{\phi}\right) = 0 \tag{9.32}$$

$$2\,\sin(\theta)\cos(\theta)\dot{\theta}\,\dot{\phi} + \sin^2(\theta)\,\ddot{\phi} = 0 \tag{9.33}$$

$$\sin(\theta)\,\ddot{\phi} = -2\,\cos(\theta)\dot{\theta}\,\dot{\phi} \tag{9.34}$$

The final step involved dividing by $\sin\theta$, which is zero at the pole $\theta = 0$. Note also that $\dot{\phi}$ is not defined at the pole.

Chapter 9 Problem 5.7

Prove a particle constrained to move on the surface $f(x, y, z) = 0$, but with no other forces, moves along geodesics. She provides the hint that the potential energy is constant (since the forces of constraint are always normal to the motion of the particle).

First let's understand the constant potential. The change in potential $dV = \mathbf{F_c} \cdot \mathbf{dr}$ where $\mathbf{F_c}$ is the sum of the conservative forces. This is explained in Chapter 6, section 8, on pp. 260 to 261. Here the only forces are the forces of constraint, and so $dV = 0$, so $V =$constant. Since the constant is arbitrary we can conveniently choose it as zero.

The Lagrangian only contains the kinetic energy,

$$L = \frac{1}{2}m\,v^2 - V = \frac{1}{2}m\,\dot{s}^2.$$

Hamilton's principle is that the particle's path is always such that the time-integral of L is stationary,

$$\int L\,dt = \frac{1}{2}m\int \dot{s}^2\,dt = \frac{1}{2}m\int \dot{s}\,ds$$

where the endpoints of the integration correspond to the origin and endpoint of the particle path. We know from Newtonian mechanics (I'm not clear that she intended us to use this here) that only forces that do work on the particle can change its KE, and therefore its speed $\dot{s}$. This point allows us to make the critical step of writing,

$$\int L\,dt = \frac{1}{2}m\,\dot{s}\int ds$$

Recall a geodesic has minimum path length between two specified endpoints, which is found by making the following integral stationary:

$$\int ds$$

So, as Boas suggested, applying Hamilton's principle to finding the particle path and the problem of finding a geodesic leads to identical problems in this case.

I find this problem somewhat artificial because it's hard to imagine a case where gravity is not important. On the other hand, it's a very good problem because if you first go off in the wrong direction, like I did, then you are confronted with dealing with constraints. This is a nice primer to the next section of this chapter.

Chapter 9 Problem 5.8

Here we learn to "eliminate the constraint equation".

The kinetic energy is just the sum of the kinetic energies of the two masses:

$$\mathrm{KE} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)$$

The potential only changes because of the movement of the suspended mass:

$$V = m\,g\,z, \quad 0 \le z \le -l$$

where z is measured positive upwards and we're assuming that the first mass is always on the surface and the second is always suspended below. The constraint is then expressed as:

$$l = r + |z| = r - z,$$

This implies that we have only two independent variables, and we must eliminate the constraint equation by eliminating either r or z. The latter gives,

$$L(t, r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{r}^2 \right) - m\,g\,(r - l) \tag{9.35}$$

The equations of motion are then derived from Lagrangian's equations. In the θ direction:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{9.36}$$

$$\frac{d}{dt}\left(m\,r^2\,\dot{\theta}\right) = 0 \tag{9.37}$$

$$m\left(2r\,\dot{r}\,\dot{\theta} + r^2\,\ddot{\theta}\right) = 0 \tag{9.38}$$

$$r\,\ddot{\theta} = -2\,\dot{r}\,\dot{\theta} \tag{9.39}$$

$$\tag{9.40}$$

where the final step assumes $r > 0$.

In the radial direction:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{9.41}$$

$$\frac{d}{dt}\left(m\,2\,\dot{r}\right) + m\,g = 0 \tag{9.42}$$

$$\ddot{r} = -\frac{g}{2} \tag{9.43}$$

Interpretation:

Note that Eqn. 9.37 is a statement of conservation of angular momentum. And the result Eqn. 9.39 implies an angular acceleration in the direction of rotation when the second mass lowers. This is intuitive, and corresponds qualitatively to a figure skater pulling in her arms. The result Eqn. 9.43 is apparent from a force diagram. The gravitational force on the second mass must accelerate two masses, and hence the vertical acceleration is half that of a free mass.

Chapter 9 Problem 5.9

Here is another problem in which we must "eliminate the constraint equation". Comparing this with the previous problem gives an intuitive feel for the dynamics of motion on a cone.

We work in cylindrical co-ordinates, and the kinetic energy is just:

$$\mathrm{KE} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right)$$

The potential is:

$$V = m\, g\, z,$$

where z is measured positive upwards. The constraint is expressed as:

$$r = z.$$

Eliminating the equation of constraint we find the Lagrangian is

$$L(t, r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{r}^2 \right) - m\, g\, r$$

Comparing this with the Lagrangian of the previous problem (Eqn. 9.35) we note that they differ by an unimportant constant. Thus the equations of motion are identical.

Interpretation:

The motion of the particle on the cone, like that in the previous problem, is constrained in the angular direction by conservation of angular momentum. In the radial direction, the acceleration is reduced because the gravitational force is oblique to the motion of the particle. It's interesting that the two different problems have identical dynamics.

Chapter 9 Problem 5.10

Approach Example 3 using cylindrical coordinates. This problem involves eliminating two equations of constraint. Otherwise it seems to be a straightforward extension of other problems of the section.

For the Lagrangian we must sum the contributions from the different masses. For the KE we have:

$$\text{KE} = \frac{1}{2}\, m_1 \left(\dot{r}^2 + r^2\,\dot{\theta}^2 + \dot{z}_1^2\right) + \frac{1}{2}\, m_2\, \dot{z}_2^2$$

while the PE is

$$V = m_1\, g\, z_1 + m_2\, g\, z_2.$$

The constraints are:

$$\tan 30 = \frac{r}{z_1} = \frac{1}{\sqrt{3}}$$

and

$$\frac{r}{\sin 30} + |z_2| = l = 2\,r - z_2$$

where $0 \leq z_2 \leq -l$, and z_1 and z_2 are measured positive upwards. Thus the Lagrangian becomes, once the z_1 and z_2 are eliminated,

$$L(r, \theta, \dot{r}, \dot{\theta}) = \text{KE} - V = \frac{1}{2}\, m_1 (4\,\dot{r}^2 + r^2\,\dot{\theta}^2) + 2\, m_2\, \dot{r}^2 - \sqrt{3}\, g\, m_1\, r - g\, m_2\, (2\,r - l)$$

Lagrange's equations give the following equations of motion:

$$\frac{d}{dt}\left(m_1\,4\,\dot{r} + m_2\,4\,\dot{r}\right) - m_1\,\dot{\theta}^2\,r + g\,\sqrt{3}\,m_1 + 2\,g\,m_2 = 0,$$

or

$$(m_1\, + m_2)\,\ddot{r} = \frac{m_1}{4}\,\dot{\theta}^2\,r - \frac{\sqrt{3}}{4}g\,m_1 - \frac{1}{2}\,g\,m_2$$

And,

$$\frac{d}{dt}\left(m_1\,r^2\,\dot{\theta}\right) = 0, \tag{9.44}$$

$$r^2\,\dot{\theta} = \text{const.} \tag{9.45}$$

These results can be compared to the solution she finds on p. 400. Because $\rho = 2\,r$, we see the results are identical.

Chapter 9 Problem 5.11

I suppose the point of this problem is to show that even complicated looking problems can sometimes be quite simple using Hamilton's principle. The only trick is to equate the vertical and angular velocity:

$$\dot{z} = a\,\dot{\theta}$$

where z is positive downwards, and θ is positive clockwise, and a is the inner radius. Perhaps the easiest way to see this is to note that for each complete rotation through 2π of the yo-yo, z increases by the perimeter, $2\pi a$. Therefore, $z = z_0 + 2\pi a(\theta/(2\pi)) = z_0 + a\,\theta$. Of course we're ignoring the change in inner radius as the yo-yo unwinds. (We'd have to be given much more information to take this into account.) The potential energy is only that due to the work against gravity. So the Lagrangian is

$$L = \frac{1}{2}\,m\,\dot{z}^2 + \frac{1}{2}\,I\,\dot{\theta}^2 - (-m\,g\,z)$$

Eliminating the equation of constraint,

$$L = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} \frac{I}{a^2} \dot{z}^2 + m g z$$

The Lagrange equation of motion follows immediately:

$$\ddot{z} = \frac{g}{1 + \frac{I}{m a^2}}$$

Interpretation: The acceleration downward under the action of gravity is reduced due to the necessity to unwind the yo-yo. The amount the acceleration is reduced goes to zero as the moment of inertia goes to zero, (*i.e.* acceleration goes to g as I goes to zero), and conversely becomes the dominant factor when $I \gg m a^2$.

9.6 Isoperimetric Problems

In the previous section, we had constraints on the variables, which could be dealt with be eliminating one of the variables. But now we have constraints on integral quantities, and another method must be used.

9.6.1 Problems

Chapter 9 Problem 6.1

Minimize the surface of revolution for a string of fixed length between two points (x_1, y_1) and (x_2, y_2).

$$\text{Area} = \int_1^2 2\,\pi\, y(x)\, ds = 2\,\pi \int_{y_1}^{y_2} y(x)\, \sqrt{1 + x'^2}\, dy$$

I'm assuming $x(y)$ is single valued. The constraint is

$$l = \int_1^2 2\, ds = \int_{y_1}^{y_2} \sqrt{1 + x'^2}\, dy$$

Thus we want the following integral to be stationary:

$$\int_{y_1}^{y_2} (2\,\pi\,y(x) + \lambda)\,\sqrt{1 + x'^2}\,dy$$

Euler's equation gives:

$$\frac{\partial}{\partial x'}\left((2\,\pi\,y(x) + \lambda)\,\sqrt{1 + x'^2}\right) = c_1$$

which eventually leads to

$$dx = \pm\frac{c_1}{\sqrt{(2\pi y + \lambda)^2 - c_1^2}}dy$$

Stuck on this integral.

One can get to the same point by using

$$F + \lambda G - y'\frac{\partial F + \lambda G}{\partial y'} = c_1$$

(see Chapter 6 problem 8.1).

Chapter 9 Problem 6.2

Chapter 10

Coordinate Transformations: Tensor Analysis

Bibliography

Boas, M. L., 1983: *Mathematical methods in the physical sciences*. John Wiley and Sons, 793 pp.