Lecture 11

Orthogonality of particle displacement vectors

In our analysis of motion in homogenous anisotropic solids, we showed that there always exist three types of motion. We will now show that they are always mutually orthogonal. For this we will start with the following Christoffel equation

$$\Gamma_{ij} u_j = \rho \left(\frac{\omega}{k} \right)^2 u_i \, ,$$

we assume that for a given propagation direction $\hat{l}$ the three solutions are

$$(u_i)_1 \quad \textit{with slowness} \; \left(\frac{\omega}{k} \right)_1 \; ,$$
$$(u_i)_2 \quad \textit{with slowness} \; \left(\frac{\omega}{k} \right)_2 \; , and$$
$$(u_i)_3 \quad \textit{with slowness} \; \left(\frac{\omega}{k} \right)_3 \; .$$

We take one of the solutions, say, *a* and write into the Christoffel equation to get

$$\Gamma_{ij} \left(u_j \right)_a = \rho \left(\frac{\omega}{k} \right)_a^2 (u_i)_a \quad ,$$

and dot both sides of the above equation with another solution, say, *b* to obtain

$$(u_i)_b \, \Gamma_{ij} \left(u_j \right)_a = \rho \left(\frac{\omega}{k} \right)_a^2 (u_i)_b (u_i)_a \quad .$$

Repetition of the same procedure with the roles of *a* and *b* interchanged, we obtain

$$(u_i)_a \Gamma_{ij} (u_j)_b = \rho \left(\frac{\omega}{k}\right)_b^2 (u_i)_a (u_i)_b \quad .$$

Now recall that Γ is symmetric. Hence the subscripts *i* and *j* in the left hand side of the above equation can be interchanged. Therefore we obtain

$$\rho \left[\left(\frac{\omega}{k}\right)_b^2 - \left(\frac{\omega}{k}\right)_a^2 \right] (u_i)_b (u_i)_a = \rho \left[\left(\frac{\omega}{k}\right)_b^2 - \left(\frac{\omega}{k}\right)_a^2 \right] \mathbf{u}_b \mathbf{u}_a = 0 \quad .$$

Thus if the phase velocities are different, the particle displacements are orthogonal. When the two phase velocities are equal, the solutions *a* and *b* can be combined into solutions that are invariant, if they are not already so.