

LECTURE 2

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2.1 Transformation of elastic coefficient matrix on orthogonal transformations

We restrict ourselves to the Cartesian system and consider the transformation of the elastic coefficient matrix on orthogonal coordinate transformation. We first start with the two-dimensional case.

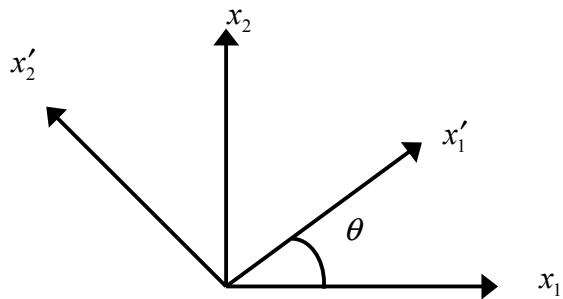


Figure 2.1 Coordinate transformation

Let (x_1, x_2) be a two-dimensional system, in terms of which an arbitrary vector \mathbf{v} can be expressed in component form as:

$$\mathbf{v} = v_1 \hat{x}_1 + v_2 \hat{x}_2 \quad . \quad (2.1)$$

Now rotate the coordinate system by some arbitrary angle (Fig. 2.1) to define a new coordinate system. In the new system, the vector \mathbf{v} can be expressed as

$$\mathbf{v} = v'_1 \hat{x}'_1 + v'_2 \hat{x}'_2 \quad . \quad (2.2)$$

It is straightforward to show that the components \mathbf{v} are related by the following equation

$$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \quad (2.3)$$

Thus the components in the primed system are related to those in the unprimed system through a transformation matrix \mathbf{A} whose elements are the direction cosines of primed system to unprimed system. Thus following the notations in Fig. (2.1), we have

$$\mathbf{v} = \mathbf{A}\mathbf{v}',$$

$$\mathbf{A} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (2.4)$$

Since \mathbf{v} is arbitrary, this can be inverted to give

$$\mathbf{v} = \mathbf{A}^{-1}\mathbf{v}',$$

$$\mathbf{A}^{-1} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \mathbf{A}^T. \quad (2.5)$$

Therefore the transformation matrix \mathbf{A} is orthogonal and that is why such a transformation is always called orthogonal transformation.

Such transformation can easily be extended to three dimensions, where \mathbf{A} becomes a 3X3 matrix. Now if we consider a tensor \mathbf{V} of order two, it is straightforward to show that

$$V'_{ij} = A_{ik} A_{jl} V_{kl} \quad . \quad (2.6)$$

Similarly for a tensor \mathbf{B} of order three, we have

$$B'_{ijk} = A_{ip} A_{jq} A_{kr} B_{pqr} \quad . \quad (2.7)$$

For a tensor \mathbf{C} of order four, we have

$$C'_{ijkl} = A_{ip} A_{jq} A_{kr} A_{ls} C_{pqrs} \quad , \quad (2.8)$$

and so on. Thus it is quite straightforward to transform tensor components from one coordinate system to another under orthogonal coordinate transformations.

We now go back to the stress-strain relation in condensed form and derive expressions for transformation of the matrix \mathbf{C} under coordinate rotations. To do this, we must take a look at the transformation of the stress vector \mathbf{t} and the strain vector \mathbf{e} . In order to do that we need to examine the transformation of the stress tensor and the strain tensor under orthogonal transformations. We know that

$$\tau'_{ij} = A_{ik} A_{jl} \tau_{kl} \quad . \quad (2.9)$$

Using the above relation we can relate each component of the stress vector in the primed system with that in the unprimed system resulting in the following equation

$$\begin{pmatrix} \tau'_{11} \\ \tau'_{22} \\ \tau'_{33} \\ \tau'_{23} \\ \tau'_{31} \\ \tau'_{12} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{pmatrix}, \quad (2.10)$$

where,

$$\mathbf{M} = \begin{pmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 & 2A_{12}A_{13} & 2A_{13}A_{11} & 2A_{11}A_{12} \\ A_{21}^2 & A_{22}^2 & A_{23}^2 & 2A_{22}A_{23} & 2A_{23}A_{21} & 2A_{21}A_{22} \\ A_{31}^2 & A_{32}^2 & A_{33}^2 & 2A_{32}A_{33} & 2A_{33}A_{31} & 2A_{31}A_{32} \\ A_{21}A_{31} & A_{22}A_{32} & A_{23}A_{33} & A_{22}A_{33} + A_{23}A_{32} & A_{21}A_{33} + A_{23}A_{31} & A_{22}A_{31} + A_{21}A_{32} \\ A_{31}A_{11} & A_{32}A_{12} & A_{33}A_{13} & A_{12}A_{33} + A_{13}A_{32} & A_{13}A_{31} + A_{11}A_{33} & A_{11}A_{32} + A_{12}A_{31} \\ A_{11}A_{21} & A_{12}A_{22} & A_{13}A_{23} & A_{12}A_{23} + A_{13}A_{22} & A_{13}A_{21} + A_{11}A_{23} & A_{11}A_{22} + A_{12}A_{31} \end{pmatrix}. \quad (2.11)$$

Similarly we obtain transformations of strain given by

$$\varepsilon'_{ij} = A_{ik} A_{jl} \varepsilon_{kl} \quad . \quad (2.12)$$

to obtain

$$\begin{pmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{33} \\ 2\varepsilon'_{23} \\ 2\varepsilon'_{31} \\ 2\varepsilon'_{12} \end{pmatrix} = \mathbf{N} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix}, \quad (2.13)$$

where,

$$\mathbf{N} = \begin{pmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 & 2A_{12}A_{13} & 2A_{13}A_{11} & 2A_{11}A_{12} \\ A_{21}^2 & A_{22}^2 & A_{23}^2 & 2A_{22}A_{23} & 2A_{23}A_{21} & 2A_{21}A_{22} \\ A_{31}^2 & A_{32}^2 & A_{33}^2 & 2A_{32}A_{33} & 2A_{33}A_{31} & 2A_{31}A_{32} \\ 2A_{21}A_{31} & 2A_{22}A_{32} & 2A_{23}A_{33} & A_{22}A_{33} + A_{23}A_{32} & A_{21}A_{33} + A_{23}A_{31} & A_{22}A_{31} + A_{21}A_{32} \\ 2A_{31}A_{11} & 2A_{32}A_{12} & 2A_{33}A_{13} & A_{12}A_{33} + A_{13}A_{32} & A_{13}A_{31} + A_{11}A_{33} & A_{11}A_{32} + A_{12}A_{31} \\ 2A_{11}A_{21} & 2A_{12}A_{22} & 2A_{13}A_{23} & A_{12}A_{23} + A_{13}A_{22} & A_{13}A_{21} + A_{11}A_{23} & A_{11}A_{22} + A_{12}A_{31} \end{pmatrix}. \quad (2.14)$$

Thus the transformation matrix \mathbf{N} of strains is very similar to that of stress except for a shift of a factor of two from upper right hand corner to the lower left hand corner. The matrices \mathbf{M} and \mathbf{N} are called *Bond Stress and Strain transformation matrices* respectively.

Now we recall the stress-strain relation

$$\mathbf{t} = \mathbf{C}\mathbf{e}, \quad (2.15)$$

and apply the Bond stress transformation to obtain

$$\mathbf{t}' = \mathbf{M}\mathbf{t} = \mathbf{M}\mathbf{C}\mathbf{e}. \quad (2.16)$$

Now we apply the Bond strain transformation to obtain

$$\mathbf{e} = \mathbf{N}^{-1}\mathbf{e}'. \quad (2.17)$$

From (2.16) and (2.17) we obtain

$$\mathbf{t}' = \mathbf{M}\mathbf{C}\mathbf{N}^{-1}\mathbf{e} \quad (2.18)$$

Therefore, the elastic coefficient matrix under orthogonal coordinate transformation becomes

$$\mathbf{C}' = \mathbf{M}\mathbf{C}\mathbf{N}^{-1} \quad (2.19)$$

It appears that the transformation of this coefficient matrix requires that inversion of a 6X6 matrix \mathbf{N} . This, however, is not necessary, since from (2.11) and (2.14) we can easily show that

$$\mathbf{N}^{-1} = \mathbf{M}^T. \quad (2.20)$$

Thus we obtain the following simple relationship for the transformation of the elastic coefficient matrix under orthogonal coordinate transformations

$$\mathbf{C}' = \mathbf{MCM}^T \quad (2.21)$$

2.2 A general rotation in Cartesian system and the corresponding rotational matrices for the elastic coefficient matrix

We restrict ourselves to the Cartesian system and consider the transformation of the elastic coefficient matrix for coordinate rotation. We will now find a way to define a general rotation and show how the matrix \mathbf{C} can be rotated by such general rotation.

Applying successive rotations about different coordinate axes can perform any general rotation of rectangular coordinates. A standard method of doing that is as follows.

First rotate that coordinates anti-clockwise through an angle ϕ about the x_3 axis, so that the new coordinates are x'_1, x'_2 , and x_3 (Figure 2.2).

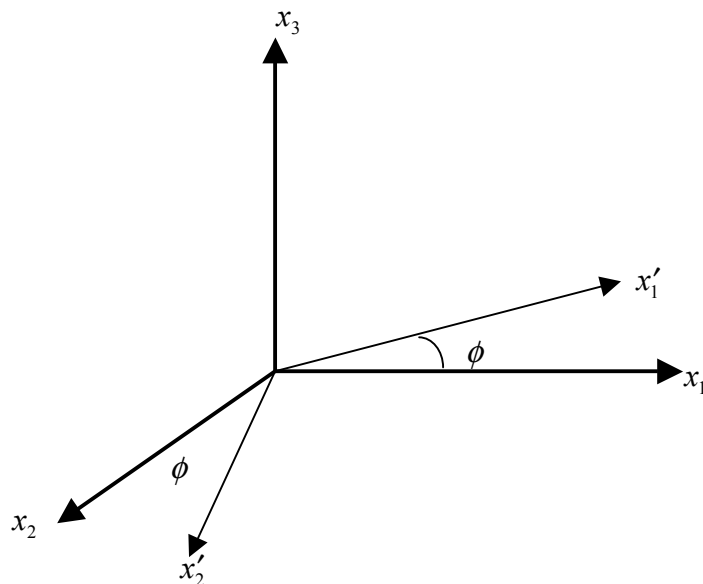


Figure 2.2. Rotation of the coordinate system counter-clock wise about the vertical axis.

The transformation matrix for such a rotation is given by

$$\mathbf{A} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.22)$$

Thus using the formula (Eq 2.11) the Bond stress transformation matrix \mathbf{M}_ϕ is given by

$$\mathbf{M}_\phi = \begin{pmatrix} \cos^2 \phi & \sin^2 \phi & 0 & 0 & 0 & -\sin 2\phi \\ \sin^2 \phi & \cos^2 \phi & 0 & 0 & 0 & \sin 2\phi \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \phi & \sin \phi & 0 \\ 0 & 0 & 0 & -\sin \phi & \cos \phi & 0 \\ \sin 2\phi/2 & -\sin 2\phi/2 & 0 & 0 & 0 & \cos 2\phi \end{pmatrix} \quad (2.23)$$

Therefore the rotated coefficient matrix for the rotation shown in Fig. (1.2) is given by

$$\mathbf{C}_\phi = \mathbf{M}_\phi \mathbf{C} \mathbf{M}_\phi^T \quad (2.24)$$

Step 2: Now rotate the new axes (x'_1, x'_2, x'_3) anti-clock wise through an angle θ about x_2' axis to go to the new axes system (x''_1, x''_2, x''_3) as shown in Figure 2.3.

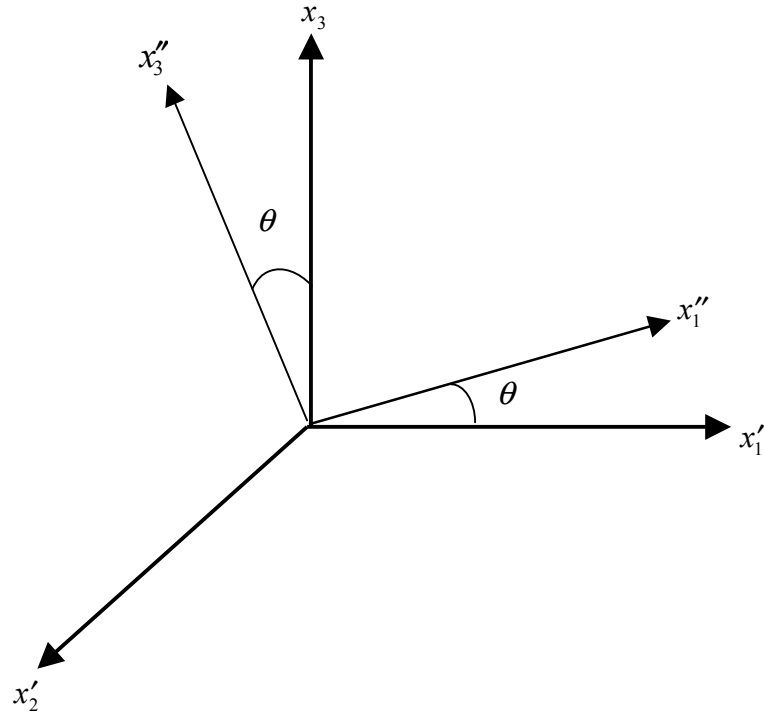


Figure 2.3. Coordinate transformation

The transformation matrix for such rotation becomes

$$\mathbf{A} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad (2.25)$$

Thus using the formula (Eq. 2.11) the Bond stress transformation matrix \mathbf{M}_θ is given by

$$\mathbf{M}_\theta = \begin{pmatrix} \cos^2\theta & 0 & \sin^2\theta & 0 & \sin 2\theta & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin^2\theta & 0 & \cos^2\theta & 0 & -\sin 2\theta & 0 \\ 0 & 0 & 0 & \cos\theta & 0 & \sin\theta \\ -\sin 2\theta/2 & 0 & \sin 2\theta/2 & 0 & \cos 2\theta & 0 \\ 0 & 0 & 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \quad (2.26)$$

Transformation \mathbf{C}_θ of $\mathbf{C}\phi$ becomes

$$\mathbf{C}_\theta = \mathbf{M}_\theta \mathbf{C}_\phi \mathbf{M}_\theta^T = \mathbf{M}_\theta \mathbf{M}_\phi \mathbf{C} \mathbf{M}_\phi^T \mathbf{M}_\theta^T. \quad (2.27)$$

Step 3: Now rotate the system (x_1'', x_2'', x_3'') by an arbitrary angle φ about x_3'' axis. Clearly in this case \mathbf{M}_θ becomes the same as \mathbf{M}_ϕ with ϕ replaced by φ . This completes the general rotation and the rotated matrix \mathbf{C}' becomes

$$\mathbf{C}' = \mathbf{M}_\varphi \mathbf{C}_\theta \mathbf{M}_\varphi^T = \mathbf{M}_\varphi \mathbf{M}_\theta \mathbf{M}_\phi \mathbf{C}_\theta \mathbf{M}_\phi^T \mathbf{M}_\theta^T \mathbf{M}_\varphi^T. \quad (2.28)$$

This can also be written as

$$\mathbf{C}' = \mathbf{M} \mathbf{C} \mathbf{M}^T, \quad (2.29)$$

where,

$$\mathbf{M} = \mathbf{M}_\varphi \mathbf{M}_\theta \mathbf{M}_\phi \quad (2.30)$$

HOMEWORK

1. Show that the scalar product of two vectors is invariant under orthogonal transformation.
2. Show that the elastic coefficient matrix for the isotropic medium is invariant under orthogonal transformation

2.3 Hexagonal Symmetry

We will now derive the elastic coefficient matrix \mathbf{C} for hexagonal symmetry, which is used to mean invariance of matrix to the rotation about one of the three coordinate axes. When this axis is taken as x_3 (x_3 vertical) the medium is often called *transversely isotropic*.

Since the elastic coefficient matrix is symmetric, it can be written as

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix}. \quad (2.31)$$

Let this be the coefficient matrix with respect to a coordinate system $\{x_1, x_2, x_3\}$. Now we rotate the coordinate system by an arbitrary angle ϕ about x_3 axis so that the new coordinate system becomes $\{x'_1, x'_2, x_3\}$. We know that the coefficient matrix \mathbf{C}' in this rotated coordinate system is given by Eq. (2.24) as

$$\mathbf{C}' = \mathbf{M}\mathbf{C}\mathbf{M}^T \quad (2.32)$$

with Eq. (2.23)

$$\mathbf{M} = \begin{pmatrix} \cos^2 \phi & \sin^2 \phi & 0 & 0 & 0 & -\sin 2\phi \\ \sin^2 \phi & \cos^2 \phi & 0 & 0 & 0 & \sin 2\phi \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \phi & \sin \phi & 0 \\ 0 & 0 & 0 & -\sin \phi & \cos \phi & 0 \\ \sin 2\phi/2 & \sin 2\phi/2 & 0 & 0 & 0 & \cos 2\phi \end{pmatrix}. \quad (2.33)$$

For the medium to be hexagonally symmetric, we have $\mathbf{C} = \mathbf{C}'$. From (2.32) and (2.33) we have

$$\mathbf{C} = \begin{pmatrix} \cos^2 \phi & \sin^2 \phi & 0 & 0 & 0 & -\sin 2\phi \\ \sin^2 \phi & \cos^2 \phi & 0 & 0 & 0 & \sin 2\phi \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \phi & \sin \phi & 0 \\ 0 & 0 & 0 & -\sin \phi & \cos \phi & 0 \\ \sin 2\phi/2 & -\sin 2\phi/2 & 0 & 0 & 0 & \cos 2\phi \end{pmatrix}$$

$$\dots \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \cos^2 \phi & \sin^2 \phi & 0 & 0 & 0 & \sin 2\phi/2 \\ \sin^2 \phi & \cos^2 \phi & 0 & 0 & 0 & -\sin 2\phi/2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \phi & -\sin \phi & 0 \\ 0 & 0 & 0 & \sin \phi & \cos \phi & 0 \\ -\sin 2\phi & \sin 2\phi & 0 & 0 & 0 & \cos \end{pmatrix} \quad (2.34)$$

This leads to 21 linear equations, which after some algebraic manipulations give the following results

$$\begin{aligned} C_{34} &= C_{35} = 0, \\ C_{44} &= C_{55}, \\ C_{45} &= 0, \\ C_{13} &= C_{23}, \\ C_{36} &= 0, \\ C_{14} &= C_{24} = C_{46} = C_{15} = C_{25} = C_{36} = 0, \\ C_{11} &= C_{23}, \\ C_{12} &= C_{11} - 2C_{66}, \\ C_{16} &= C_{26} = 0. \end{aligned} \quad (2.35)$$

Thus the elastic coefficient matrix for the hexagonal symmetry is given by the following

$$C = \begin{pmatrix} C_{11} & C_{11} - 2C_{66} & C_{13} & 0 & 0 & 0 \\ C_{11} - 2C_{66} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}. \quad (2.36)$$

Therefore we require only five elastic constants $(C_{11}, C_{13}, C_{33}, C_{44}, C_{66})$ to describe the elastic coefficient matrix in a transversely isotropic medium.

2.4 Positive definite conditions

Requirement of positive definite strain energy is

$$C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq 0, \dots (= 0 \text{ iff } \dots \varepsilon_{ij} \equiv 0) \quad (2.37)$$

Thus we have

$$C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = C_{IJ} \varepsilon_I \varepsilon_J > 0. \quad (2.38)$$

This means that the 6X6 matrix \mathbf{C} is a positive definite. For hexagonal symmetry this requires that

$$\begin{aligned} C_{66} &> 0, \\ C_{44} &> 0, \\ C_{33} &> 0, \\ C_{11} C_{33} &> C_{13}^2, \\ C_{11} C_{33} &> C_{13}^2 + C_{33} C_{66}. \end{aligned} \quad (2.39)$$

or,

$$\begin{aligned} C_{33}(C_{11} - C_{66}) &> C_{13}^2, \\ C_{11} &> C_{66}. \end{aligned} \tag{2.40}$$

If we compare with isotropy Eq. (1.10) we obtain

$$\begin{aligned} \mu &> 0, \\ \lambda + 2\mu &> 0, \\ (\lambda + 2\mu)(\lambda + 2\mu) &> \lambda^2, \\ 3\mu\lambda + 2\mu^2 &> 0, \\ \left(\lambda + \frac{2}{3}\mu\right) &= k(\text{bulk modulus}) > 0. \end{aligned} \tag{2.41}$$