The CONCEPT of the EFFECTIVE MEDIUM (Helbig, 1998)

The cause of seismic anisotropy is always some internal structure on a scale that is small to the resolution of the method applied. This does even include the anisotropy of crystals: the internal structure is the rearrangement of the ions on a 3D-lattice on a scale that is small compared to optical wavelengths (though not small to the wavelengths of x-rays).

Internal structures in geological media that can lead to anisotropy are:

- Oriented cracks
- Lamination (periodic sequences of thin layers)
- Parallel fractures
- Oriented grains
- Clay (orientation of plate-like minerals)

In many seismic observations (in seismic frequency band) that does not resolve the elements of the internal structure, these media appear to be transversely isotropic. One often calls the anisotropy of an effective medium "apparent anisotropy" or "quasi-anisotropy" to distinguish it from "intrinsic anisotropy." This distinction is , however, not necessary as the anisotropy due to crystals is also due to ordered internal structure of the ions.

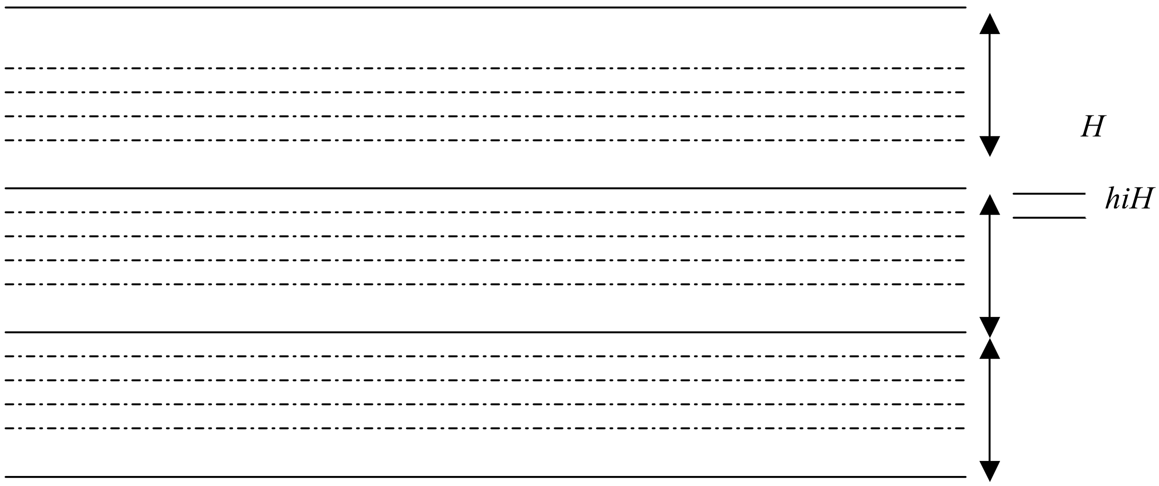
In addition to material anisotropy, there are conditions where the propagation of waves is anisotropic even if the medium is isotropic. Such conditions could be moving coordinate systems such as the propagation of sound in an air or water medium (wind, current) either at rest observed from a fixed platform, or in a moving medium (wind, ocean) observed from a fixed platform. The propagation of electromagnetic waves in plasmas becomes anisotropic under the influence of external magnetic field.

Anisotropy due to periodic layering

Geophysical media often exhibit anisotropic behavior due to alternating strata of material, each stratum itself being isotropic. This can occur over a range of length scales. *When probed with radiation of a wavelength much larger than width of the strata, such stratified regions exhibit material properties that appear to be space-independent but direction-dependent depending on the angle with respect to the axis perpendicular to the plane of stratification.* Such a medium is said to be transversely isotropic and has five independent elastic moduli. Here I will closely follow Schoenberg (1983) to derive the equivalent TI parameters based on the quasi-static considerations as outlined in Helbig (1968). Schoenberg (1983) also derived an exact formulation for periodic media using propagator matrices but I will not discuss that here.

We will now follow Schoenberg (1983) to derive the elastic behavior of periodically arranged fine layers. Let z be the axis perpendicular to layering and let the period of layering is *H*. Assume that one period is made up of *N* homogeneous layers, each with shear modulus μ_i. Poisson's ratio ν_i, and thickness h_iH, $i=1,2,3,\ldots,N$. Let γ_i be defined as the ratio of shear speed β_i to compressional wave speed α_i so that

$$\gamma_i = \frac{\beta_i^2}{a_i^2} = \frac{\frac{1}{2} - \nu_i}{1 - \nu_i} . \tag{1}$$



H
hiH

For stress and strain fields whose scale of variation is much greater than H, effective transversely isotropic modulii can be derived in terms of the μ_i, γ_i, and h_i. There are five independent elastic constants:

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix} \tag{2}$$

where $C_{12} = C_{11} - 2C_{66}$.

Now we make the following arguments:

- Under quasi-static or low frequency equilibrium requirement, the field is assumed to vary slowly with respect to *H*, the spatial period of layer aggregate. That is, the stresses that act on a face perpendicular to the x_3 axis, i.e., τ_{13}, τ_{23}, and τ_{33} are assumed constant (continuous) across a set of layers of width *H* while τ_{11}, τ_{22}, and τ_{12} are discontinuous. $\varepsilon_{33,i}$
- The layers are in welded contact. Since all the layers are constrained to have some deformation in the x_1x_2 plane, strains that lie in a plane parallel to the layering,

i.e., ε_{11}, ε_{22}, and ε_{12} are assumed constant (continuous). This is because the layers are constrained to having the same in-plane motion in a medium of infinite extent in the x_1 and x_2 directions. The ε_{33}, $\varepsilon_{23,}$ and ε_{13} are discontinuous.

The ε_{33}, $\varepsilon_{23,}$ and ε_{13} components of strain over a full spatial period H can be written in terms of the strains of the individual layers. For example, let the displacement components be denoted by u_1, u_2, and u_3. The strain ε_{33}^{i} for the layer i is given

$$\varepsilon_{33}^{i} = \frac{u_{3i}^{bottom} - u_{3i}^{top}}{h_i H} = \frac{\Delta u_{3i}}{h_i H} \quad .$$

The average strain ε_{33} over a full spatial period is given by

$$\varepsilon_{33} = \frac{u_3\left(x_3 + H\right) - u_3\left(x_3\right)}{H} = \frac{1}{H}\sum_{i=1}^{N}\Delta u_{3i} = \frac{1}{H}\sum_{i=1}^{N} H h_i \varepsilon_{33}^{i} = < \varepsilon_{33} > \ , \qquad (3)$$

where <> denotes a thickness weighted average.

Similarly, the average strains $\varepsilon_{23,}$ and ε_{13} , across a full period can be found, using displacements u_2, and u_3 respectively, to be given by

$$\varepsilon_{23} = \frac{u_2\left(x_3 + H\right) - u_2\left(x_3\right)}{H} = \frac{1}{H}\sum_{i=1}^{N}\Delta u_{2i} = \frac{1}{H}\sum_{i=1}^{N} h_i \varepsilon_{23}^{i} = < \varepsilon_{23} > \ , \qquad (4)$$

and

$$\varepsilon_{31} = \frac{u_1\left(x_3 + H\right) - u_1\left(x_3\right)}{H} = \frac{1}{H}\sum_{i=1}^{N} \Delta u_{1i} = \frac{1}{H}\sum_{i=1}^{N} h_i \varepsilon_{31}^i = < \varepsilon_{31} > \ . \qquad (5)$$

The in-plane average stresses may be derived in a similar way. If f_{12}^i the *force per unit length* in the x_1-direction on an x_2 face, the stress

$$\tau_{12}^i = \frac{f_{12}^i}{h_i H} \ .$$

The average stress across the full width

$$\tau_{12} = \frac{1}{H}\sum_{i=1}^{N} f_{12}^i = \frac{1}{H}\sum_{i=1}^{N} h_i H \tau_{12}^i = \sum_{i=1}^{N} h_i \tau_{12}^i = < \tau_{12}^i > \ . \qquad (6)$$

Similarly we have

$$\tau_{11} = \frac{1}{H}\sum_{i=1}^{N} f_{11}^i = \frac{1}{H}\sum_{i=1}^{N} h_i H \tau_{11}^i = \sum_{i=1}^{N} h_i \tau_{11}^i = < \tau_{11}^i >, \qquad (7)$$

and

$$\tau_{22} = \frac{1}{H}\sum_{i=1}^{N} f_{22}^i = \frac{1}{H}\sum_{i=1}^{N} h_i H \tau_{22}^i = \sum_{i=1}^{N} h_i \tau_{22}^i = < \tau_{22}^i > . \qquad (8)$$

Now, consider the relations between shear stress and shear strain. In each layer

$$\begin{aligned}
\tau_{23} &= \tau_{23}^{i} = 2\mu_i \varepsilon_{23}^{i} \\
\tau_{31} &= \tau_{31}^{i} = 2\mu_i \varepsilon_{31}^{i} \qquad (9) \\
\tau_{12}^{i} &= 2\mu_i \varepsilon_{12}^{i} = 2\mu_i \varepsilon_{12}
\end{aligned}$$

From the first of Eq (9) we have

$$\varepsilon_{23}^{i} = \frac{\tau_{23}}{2\mu_i},$$

or

$$\begin{aligned}
\varepsilon_{23} &= <\varepsilon_{23}^{i}> = <\mu^{-1}> \frac{\tau_{23}}{2} \\
\varepsilon_{31} &= <\varepsilon_{31}^{i}> = <\mu^{-1}> \frac{\tau_{31}}{2} \qquad . (10) \\
\tau_{12} &= 2<\mu>\varepsilon_{12}
\end{aligned}$$

Comparison between Eq. (10) and Eq. (2) reveals that

$$\boxed{C_{44} = 1/<\mu^{-1}>, C_{66} = <\mu>}$$

Now consider the relation between normal stress τ_{33} and normal strains (using isotropic Hooke's law). In each layer

$$\tau_{33}=\tau_{33}^{i}=\frac{\left[(1-2\gamma_i)(\varepsilon_{11}+\varepsilon_{22})+\varepsilon_{33}^{i}\right]\mu_i}{\gamma_i}, \tag{11a}$$

or

$$\tau_{33}=(1-2<\gamma>)<\frac{\gamma}{\mu}>^{-1}(\varepsilon_{11}+\varepsilon_{22})+<\frac{\gamma}{\mu}>^{-1}\varepsilon_{33}\,. \tag{11b}$$

Comparison between (11b) and (2) reveal that

$$\boxed{C_{33}=<\gamma/\mu>^{-1}\ ,\ C_{13}=(1-2<\gamma>)<\gamma/\mu>^{-1}}$$

Now consider finally the relation between the normal stress τ_{11}, and the normal strains. In each layer, we have

$$\tau_{11}^{i}=\ \tau_{11}^{i}=\left[\varepsilon_{11}+(1-2\gamma_i)\left(\varepsilon_{22}+\varepsilon_{33}^{i}\right)\right]\frac{\mu_i}{\gamma_i}\,. \tag{12}$$

From Eq. 11(a) and (12) we have

$$\tau_{11}^{i} = \left[\varepsilon_{11} + (1-2\gamma_i)\varepsilon_{22}\right]\frac{\mu_i}{\gamma_i} + (1-2\gamma_i)\left[\tau_{33} - (1-2\gamma_i)(\varepsilon_{11}+\varepsilon_{22})\frac{\mu_i}{\gamma_i}\right]$$
$$= 2\mu_i\varepsilon_{11} + 2(1-2\gamma_i)\mu_i(\varepsilon_{11}+\varepsilon_{22}) + (1-2\gamma_i)\tau_{33} \qquad (13)$$

Averaging and substituting for τ_{33} from Eq.(12) into Eq. (11) we have

$$\tau_{11} = 2<\mu>\varepsilon_{11} + \left[2<\mu> - 4<\gamma\mu> + (1-2<\gamma>)^2 <\gamma/\mu>^{-1}\right](\varepsilon_{11}+\varepsilon_{22})$$
$$(1-2<\gamma>)<\gamma/\mu>^{-1}\varepsilon_{33} \qquad (14)$$

Comparison between (14) and (2) reveals that

$$\boxed{C_{11} = 4<\mu> - 4<\gamma\mu> + (1-2<\gamma>)^2 <\gamma/\mu>^{-1}}$$

Thus we were able to express all five coefficients in terms of the elastic parameters of the constituent layers.

HOMEWORK: Show that for the TI medium that is a long-wavelength equivalent of fine periodic layers, the $C_{66} \geq C_{44}$

SOLUTION TO HOMEWORK PROBLEM

Prove that $C_{66} \geq C_{44}$ for transversely isotropic media generated by fine isotropic layering

We know that $C_{66} = < \mu >$ and $C_{44} = \dfrac{1}{< 1/\mu >}$, where

$< \mu > = \sum_{i=1}^{N} h_i \mu_i$ and $< 1/\mu > = \sum_{i=1}^{N} \dfrac{h_i}{\mu_i}$ with $\sum_{i=1}^{N} h_i = 1$.

Thus $C_{66} \geq C_{44}$ means $< \mu > \geq \dfrac{1}{< 1/\mu >}$ or

$$< \mu > < 1/\mu > \geq 1 \quad . \qquad (1)$$

In other words, we need to prove (1).

Recall that for any numbers $a_1, a_2, a_3, a_4, \ldots\ldots, a_n$ and $b_1, b_2, b_3, b_4, \ldots\ldots, b_n$ we have:

$$\begin{aligned} &\left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\left(b_1^2 + b_2^2 + \ldots + b_n^2\right) - \left(a_1 b_1 + a_2 b_2 + \ldots + a_n b_2\right)^2 \\ &\qquad = \left(a_1 b_2 - a_2 b_1\right)^2 + \left(a_1 b_3 - a_3 b_1\right)^2 + \ldots \\ &\qquad \geq 0 \end{aligned} .$$

Therefore, in general

$$\sum a_i^2 \sum b_i^2 \geq \left(\sum a_i b_i\right)^2 . \qquad (2)$$

Now let

$$a_i = \sqrt{h_i\, \mu_i} \qquad \text{and} \qquad b_i = \sqrt{\frac{h_i}{\mu_i}} \quad .$$

This is possible since μ_i is always positive and so is h_i. Therefore

$$\left(\sum_{i=1}^{N} h_i \mu_i\right)\left(\sum_{i=1}^{N} \frac{h_i}{\mu_i}\right) \geq \left(\sum_{i=1}^{N} \sqrt{h_i \mu_i}\sqrt{\frac{h_i}{\mu_i}}\right)^2 = \left(\sum_{i=1}^{N} h_i\right)^2 = 1 \quad .$$

or,

$$\boxed{<\mu><\frac{1}{\mu}> \geq 1}$$

Q.E.D