EFFECTIVE MEDIUM: A GENERAL DERIVATION

In the previous section we followed Schoenberg (198x) to derive an expression for an equivalent TI parameters for periodically stratified isotropic layers. Here we will closely follow Schoenberg and Douma (1988) and derive a generalized expression for long wavelength anisotropy due to

- Periodic anistropic layers , and
- Parallel fratures and aligned cracks.

Consider a stratified medium made up of perfectly bounded homogeneous, but not necessarily isotropic layers. Let the x_3-axis be perpendicular to the layering and assume that there are *n* different constituent layers, arranged so that in each sufficiently large interval one finds the same proportion of each medium, e.g., a periodic sequence of layers. Each anisotropic constituent has a relative thickness, h_i, i=1,2, . . ., n so that $h_1 + h_2 + ... + h_n = 1$, a density ρ_i and an elastic modulus tensor c^i_{qprs} relating sress τ^i_{pq} to strain ε^i_{rs}. In condensed notation we have

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} = \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix}$$

The elastic moduli of the equivalent anisotropic medium, in the long wavelength (quasi-static) limit, due to a layered medium, composed of anisotropic constituent layers can be expressed in terms of thickness-weighted averages of functions of the moduli of the constituents.

- The long-wavelength assumption on stress is that all stress components acting on surfaces parallel to the layering are the same in all the layers, i.e.,

  $$\tau_{33}^i = \tau_3^i = \tau_3 \quad \tau_{23}^i = \tau_4^i = \tau_4 \quad \tau_{13}^i = \tau_5^i = \tau_5 \quad .$$

- The long wavelength kinematic assumption is that over many layers, the layers move together (so that derivatives of in-plane displacements with respect to in-plane coordinates, x_1 and x_2 are the same) implying that all strain components lying in the plane of the layering are the same in all layers, i.e.,

  $$\varepsilon_{11}^i = \varepsilon_1^i = \varepsilon_1 \quad \varepsilon_{22}^i = \varepsilon_2^i = \varepsilon_2 \quad 2\varepsilon_{12}^i = \varepsilon_6^i = \varepsilon_6 \quad .$$

The other stress and strain components may vary from layer to layer. In each layer such a component may be taken as its average value across the thickness of that layer. Based on thee ideas, we define the following vectors

$$\mathbf{S}_1^i = \begin{bmatrix} \tau_1^i \\ \tau_2^i \\ \tau_6^i \end{bmatrix} \qquad \mathbf{E}_2^i = \begin{bmatrix} \varepsilon_3^i \\ \varepsilon_4^i \\ \varepsilon_5^i \end{bmatrix} \qquad \text{layer dependent}$$

$$\mathbf{S}_2 = \begin{bmatrix} \tau_3 \\ \tau_4 \\ \tau_5 \end{bmatrix} \qquad \mathbf{E}_1 = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$ layer independent.

These allow the stress-stain relations in any layers to be written as

$$\mathbf{S}_1^i = \mathbf{M}^i \mathbf{E}_1 + \mathbf{P}^i \mathbf{E}_2^i \tag{3(a)}$$

$$\mathbf{S}_2 = \left(\mathbf{P}^i\right)^T \mathbf{E}_1 + \mathbf{N}^i \mathbf{E}_2^i \tag{3(b)}$$

where

$$\mathbf{M}^i = \begin{bmatrix} c_{11}^i & c_{12}^i & c_{16}^i \\ c_{12}^i & c_{22}^i & c_{26}^i \\ c_{16}^i & c_{26}^i & c_{66}^i \end{bmatrix} \quad \mathbf{N}^i = \begin{bmatrix} c_{33}^i & c_{34}^i & c_{35}^i \\ c_{34}^i & c_{44}^i & c_{45}^i \\ c_{35}^i & c_{45}^i & c_{55}^i \end{bmatrix} \quad \mathbf{P}^i = \begin{bmatrix} c_{13}^i & c_{14}^i & c_{15}^i \\ c_{23}^i & c_{24}^i & c_{25}^i \\ c_{36}^i & c_{46}^i & c_{56}^i \end{bmatrix} \tag{4}$$

Note that $\mathbf{M}^i$ and $\mathbf{N}^i$ are symmetric matrices. From Eq. 3(b) we have

$$\begin{aligned} \left(\mathbf{N}^i\right)^{-1} \mathbf{S}_2 &= \left(\mathbf{N}^i\right)^{-1} \left(\mathbf{P}^i\right)^T \mathbf{E}_1 + \mathbf{E}_2^i \\ \mathbf{E}_2^i &= \left(\mathbf{N}^i\right)^{-1} \mathbf{S}_2 - \left(\mathbf{N}^i\right)^{-1} \left(\mathbf{P}^i\right)^T \mathbf{E}_1 \end{aligned} \tag{5}$$

From 3(a) and (5) we obtain

$$\mathbf{S}_1^i = \mathbf{M}^i \mathbf{E}_1 + \mathbf{P}^i \left[\left(\mathbf{N}^i \right)^{-1} \mathbf{S}_2 - \left(\mathbf{N}^i \right)^{-1} \left(\mathbf{P}^i \right)^T \mathbf{E}_1 \right]. \qquad (6)$$

Now, following the previous section, we let the weighted averages over all the constituent layers be denoted by <.>. From (6) we have

$$\left\langle \mathbf{S}_1 \right\rangle = \left[\left\langle \mathbf{M} \right\rangle - \left\langle \mathbf{P}\mathbf{N}^{-1}\mathbf{P}^T \right\rangle \right] \mathbf{E}_1 + \left\langle \mathbf{P}\mathbf{N}^{-1} \right\rangle \mathbf{S}_2 \qquad 7(a)$$

and from (5) we have

$$\mathbf{S}_2 = \left\langle \mathbf{N}^{-1} \right\rangle^{-1} \left\langle \mathbf{N}^{-1}\mathbf{P}^T \right\rangle \mathbf{E}_1 + \left\langle \mathbf{N}^{-1} \right\rangle^{-1} \left\langle \mathbf{E}_2 \right\rangle \quad . \qquad 7(b)$$

Finally substituting the expression for $\mathbf{S}_2$ from 7(b) into 7(a), we write the elastic moduli for the media equivalent to the stratified medium in the long wavelength limit, in matrix form as

$$\left\langle \mathbf{S}_1 \right\rangle = \mathbf{M}_e \mathbf{E}_1 + \mathbf{P}_e \left\langle \mathbf{E}_2 \right\rangle \qquad 8(a)$$

$$\mathbf{S}_2 = \mathbf{P}_e^T \mathbf{E}_1 + \mathbf{N}_e \left\langle \mathbf{E}_2 \right\rangle \quad . \qquad 8(b)$$

with

(9)

$$\boxed{\begin{array}{l} \mathbf{N}_e = \left\langle \mathbf{N}^{-1} \right\rangle^{-1} \\ \mathbf{P}_e = \left\langle \mathbf{P}\mathbf{N}^{-1} \right\rangle \mathbf{N}_e \\ \mathbf{M}_e = \left\langle \mathbf{M} \right\rangle - \left\langle \mathbf{P}\mathbf{N}^{-1}\mathbf{P}^T \right\rangle + \left\langle \mathbf{P}\mathbf{N}^{-1} \right\rangle \mathbf{N}_e \left\langle \mathbf{N}^{-1}\mathbf{P}^T \right\rangle \end{array}}$$

Thus we have the desired result in Eq. (9)! Given a stack of homogeneous anisotropic periodic layers, equivalent long wavelength anisotropic parameters can be evaluated using eq. (9).

SPECIAL CASE:

The ith constiuent is transversely isotropic with the x_3-axis as the axis of symmetry

$$\mathbf{M}_i = \begin{bmatrix} c_{11}^i & c_{11}^i - 2c_{66}^i & 0 \\ c_{11}^i - 2c_{66}^i & c_{11}^i & 0 \\ 0 & 0 & c_{66}^i \end{bmatrix} \quad \mathbf{N}_i = \begin{bmatrix} c_{33}^i & 0 & 0 \\ 0 & c_{44}^i & 0 \\ 0 & 0 & c_{44}^i \end{bmatrix} \quad \mathbf{P}_i = \begin{bmatrix} c_{13}^i & 0 & 0 \\ c_{13}^i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad .$$

Note that when the constituent layer is isotropic

$$c_{44}^i = c_{66}^i = \mu_i \quad ; c_{11}^i = c_{33}^i = \lambda_i + 2\mu_i \ ; c_{13}^i = \lambda_i \quad .$$

If all the constituent layers are transversely isotropic, the equivalent homogeneous medium is transversely isotropic and from (9) the moduli are given by

$$\mathbf{N}_e = \begin{bmatrix} c_{33} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{44} \end{bmatrix} = \begin{bmatrix} 1/\left\langle \dfrac{1}{c_{33}} \right\rangle & 0 & 0 \\ 0 & 1/\left\langle \dfrac{1}{c_{33}} \right\rangle & 0 \\ 0 & 0 & 1/\left\langle \dfrac{1}{c_{33}} \right\rangle \end{bmatrix}$$

$$\mathbf{P}_e = \begin{bmatrix} c_{13} & 0 & 0 \\ c_{13} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \left\langle \dfrac{c_{13}}{c_{33}} \right\rangle / \left\langle \dfrac{1}{c_{33}} \right\rangle & 0 & 0 \\ \left\langle \dfrac{c_{13}}{c_{33}} \right\rangle / \left\langle \dfrac{1}{c_{33}} \right\rangle & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_e = \begin{bmatrix} c_{11} & c_{11} - 2c_{66} & 0 \\ c_{11} - 2c_{66} & c_{11} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{11} - 2\langle c_{66} \rangle & 0 \\ c_{11} - 2\langle c_{66} \rangle & c_{11} & 0 \\ 0 & 0 & \langle c_{66} \rangle \end{bmatrix},$$

where

$$c_{11} = \langle c_{11} \rangle - \left\langle \frac{c_{13}^2}{c_{33}} \right\rangle + \langle c_{13} / c_{33} \rangle^2 / \left\langle \frac{1}{c_{33}} \right\rangle \quad ,$$

identical to the result of Backus (1962).

Reference

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