

## LECTURE 7

## 2.2. More on Christoffel Equation

We will now specialize to abbreviated notations using  $\mathbf{t}$  and  $\mathbf{e}$  as six component stress and strain vectors respectively and  $\mathbf{C}$  as a 6X6 elastic coefficient matrix as discussed in chapter 1 such that

$$\mathbf{t} = \mathbf{C}\mathbf{e} \quad , \quad (2.11)$$

where,

$$\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{pmatrix} = \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{pmatrix} . \quad (2.12)$$

$$\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{pmatrix} . \quad (2.13)$$

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} . \quad (2.14)$$

Also,

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} e_1 & \frac{1}{2}e_6 & \frac{1}{2}e_5 \\ \frac{1}{2}e_6 & e_2 & \frac{1}{2}e_4 \\ \frac{1}{2}e_5 & \frac{1}{2}e_4 & e_3 \end{pmatrix}. \quad (2.15)$$

From Eq. (2.13) and (2.15) we have

$$\mathbf{e} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad (2.16)$$

Similarly,

$$\nabla \cdot \boldsymbol{\tau} = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} \\ \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} \\ \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial t_1}{\partial x_1} + \frac{\partial t_6}{\partial x_2} + \frac{\partial t_5}{\partial x_3} \\ \frac{\partial t_6}{\partial x_1} + \frac{\partial t_2}{\partial x_2} + \frac{\partial t_4}{\partial x_3} \\ \frac{\partial t_5}{\partial x_1} + \frac{\partial t_4}{\partial x_2} + \frac{\partial t_3}{\partial x_3} \end{pmatrix}. \quad (2.17)$$

or,

$$\nabla \cdot \tau = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{pmatrix}. \quad (2.18)$$

Using the above equation we obtain the following form of the linearized momentum equation in indicial notation

$$\begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{55} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{55} & C_{65} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ = -\rho\omega^2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \quad (2.19)$$

Eq. (2.19) is the elastodynamic wave equation for a general anisotropic medium. Now we restrict ourselves to a homogeneous medium and consider plane waves propagating in the direction

$$\hat{l} = \hat{x}_1 l_1 + \hat{x}_2 l_2 + \hat{x}_3 l_3 \quad , \quad (2.20)$$

Such plane waves will have fields proportional to  $e^{i(k\hat{l}\cdot\mathbf{r}-\omega t)}$ ,  $\mathbf{r}$  being the position vector.

Next we apply a spatial Fourier transform to Eq. (2.19)

(Note that  $k = \frac{\omega}{c}$ ,  $kl_i = k_i$ ;  $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$  .) to obtain the following equation

$$\begin{aligned} & -k^2 \begin{pmatrix} l_1 & 0 & 0 & 0 & l_3 & l_2 \\ 0 & l_2 & 0 & l_3 & 0 & l_1 \\ 0 & 0 & l_3 & l_2 & l_1 & 0 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \\ 0 & l_3 & l_2 \\ l_3 & 0 & l_1 \\ l_2 & l_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ & = -\rho\omega^2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \end{aligned} \quad (2.21)$$

For a source free region the above equation leads to the following Christoffel equation

$$\begin{aligned} k^2 \Gamma_{ij} u_j &= \rho\omega^2 u_i \quad , \\ k^2 \Gamma \cdot \mathbf{u} &= \rho\omega^2 \mathbf{u} \quad . \end{aligned} \quad (2.22)$$

where,

$$\Gamma = \begin{pmatrix} \alpha & \delta & \varepsilon \\ \delta & \beta & \zeta \\ \varepsilon & \zeta & \gamma \end{pmatrix}, \quad (2.23)$$

with,

$$\begin{aligned}
\alpha &= C_{11}l_1^2 + C_{66}l_2^2 + C_{55}l_3^2 + 2C_{56}l_2l_3 + 2C_{15}l_3l_1 + 2C_{16}l_1l_2, \\
\beta &= C_{66}l_1^2 + C_{22}l_2^2 + C_{44}l_3^2 + 2C_{24}l_2l_3 + 2C_{46}l_3l_1 + 2C_{26}l_1l_2, \\
\gamma &= C_{55}l_1^2 + C_{44}l_2^2 + C_{33}l_3^2 + 2C_{34}l_2l_3 + 2C_{35}l_3l_1 + 2C_{45}l_1l_2, \\
\delta &= C_{16}l_1^2 + C_{26}l_2^2 + C_{45}l_3^2 + (C_{46} + C_{25})l_2l_3 + (C_{14} + C_{56})l_3l_1 + (C_{12} + C_{66})l_1l_2, \\
\varepsilon &= C_{15}l_1^2 + C_{46}l_2^2 + C_{35}l_3^2 + (C_{45} + C_{36})l_2l_3 + (C_{13} + C_{55})l_3l_1 + (C_{14} + C_{36})l_1l_2, \\
\varsigma &= C_{56}l_1^2 + C_{24}l_2^2 + C_{34}l_3^2 + (C_{44} + C_{23})l_2l_3 + (C_{36} + C_{45})l_3l_1 + (C_{25} + C_{46})l_1l_2.
\end{aligned} \tag{2.24}$$

Thus the Christoffel equation for a general anisotropic (triclinic) elastic medium is given by

$$k^2 \Gamma_{ij} u_j = \rho \omega^2 u_i \quad ,$$

$$k^2 \begin{pmatrix} \alpha & \delta & \varepsilon \\ \delta & \beta & \varsigma \\ \varepsilon & \varsigma & \gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \rho \omega^2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{2.25}$$

where  $\alpha, \beta$  etc. are given in Eq. (2.24). Also note that

$$l_1 = \frac{k_1}{k}, l_2 = \frac{k_2}{k}, l_3 = \frac{k_3}{k} \quad ,$$

are the direction cosines of the propagation direction. This result can easily be specialized to particular crystal systems by imposing constraints on the elastic coefficient matrix. [See *Auld* 1973, Appendix 2].

### 2.3. Slowness (Inverse Velocity) Surface

We start with the Christoffel equation (2.25) in the following form

$$(k^2 \Gamma_{ij} - \rho \omega^2 \delta_{ij}) u_j = 0 \quad . \quad (2.26)$$

A dispersion relation from this equation can be obtained by setting the characteristic determinant to zero

$$|k^2 \Gamma_{ij} - \rho \omega^2 \delta_{ij}| = 0 \quad . \quad (2.27)$$

At a given frequency, Eq. (2.27) defines a surface in  $k$ -space, which gives  $k$  as a function of its direction  $\hat{l}$ . This surface is called a *wave number* surface.

Since the first term in the dispersion relation is proportional to  $k^2$  and the second term to  $\omega^2$ , this relation can be expressed in terms of slowness  $p = \frac{k}{\omega}$ . The resulting equation denotes a surface in  $p$ -space, which gives  $p$  as a function of  $\hat{l}$ . This surface is called a *slowness (or inverse velocity) surface*. Once the characteristic equation (2.27) is solved in terms of its eigenvalues, they when substituted into Eq. (2.26) give eigenvectors or polarization vectors  $bf$ , giving all possible motions, which can be expected in an anisotropic elastic system. In general, numerical computations are required to carry out this procedure, but for special cases the entire calculation can be performed analytically.

The characteristic equation (2.27) when written in terms of  $p = \frac{k}{\omega}$  becomes

$$|p^2 \Gamma_{ij} - \rho \delta_{ij}| = 0, \quad (2.28)$$

Thus, in general, the characteristic equation is a cubic in  $p^2$

In general, the solution of Eq. (2.28) can be numerically found which gives three possible roots of  $p^2 = \frac{k^2}{\omega^2}$  and this as a function of direction of propagation gives slowness surfaces.  $p^2$  obtained by solving Eq. (2.28), when substituted in Eq. (2.25), gives three eigen vectors for the displacement **u**.