LECTURE 7A

2.3. *Examples of slowness surfaces*

2.3.1. Cubic Symmetry

The elastic coefficient matrix for a cubic symmetry is given as

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}. \qquad (2.30)$$

From Eq. (2.24) the components of the Christoffel matrix for cubic class are given by

$$\begin{aligned} \alpha &= C_{11}l_1^2 + C_{44}\left(1-l_1^2\right), \\ \beta &= C_{11}l_2^2 + C_{44}\left(1-l_2^2\right), \\ \gamma &= C_{11}l_3^2 + C_{44}\left(1-l_3^2\right), \\ \delta &= \left(C_{12}+C_{44}\right)l_1l_2, \\ \varepsilon &= \left(C_{12}+C_{44}\right)l_3l_1, \\ \varsigma &= \left(C_{12}+C_{44}\right)l_2l_3. \end{aligned} \qquad (2.31)$$

Substituting these into Eq. (2.25) we obtain the Christoffel equation for the cubic class as follows

$$k^2 \begin{pmatrix} C_{11} l_1^2 + C_{44}\left(1 - l_1^2\right) & \left(C_{12} + C_{44}\right) l_1 l_2 & \left(C_{12} + C_{44}\right) l_1 l_3 \\ \left(C_{12} + C_{44}\right) l_1 l_2 & C_{11} l_2^2 + C_{44}\left(1 - l_2^2\right) & \left(C_{12} + C_{44}\right) l_2 l_3 \\ \left(C_{12} + C_{44}\right) l_1 l_3 & \left(C_{12} + C_{44}\right) l_2 l_3 & C_{11} l_3^2 + C_{44}\left(1 - l_3^2\right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \rho \omega^2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \qquad (2.32)$$

This gives the characteristic equation (in p-domain) for the cubic class

$$\begin{vmatrix} p^2\left[C_{11} l_1^2 + C_{44}\left(1 - l_1^2\right)\right] - \rho & p^2\left[C_{11} + C_{44}\right] l_1 l_2 & p^2\left[C_{11} + C_{44}\right] l_1 l_3 \\ p^2\left[C_{11} + C_{44}\right] l_1 l_2 & p^2\left[C_{11} l_2^2 + C_{44}\left(1 - l_2^2\right)\right] - \rho & p^2\left[C_{11} + C_{44}\right] l_2 l_3 \\ p^2\left[C_{11} + C_{44}\right] l_1 l_3 & p^2\left[C_{11} + C_{44}\right] l_2 l_3 & p^2\left[C_{11} l_3^2 + C_{44}\left(1 - l_3^2\right)\right] - \rho \end{vmatrix} = 0. \qquad (2.33)$$

Eq. (2.33) needs to be evaluated numerically. However, general features of the dispersion relation can be deduced by considering special propagation directions, for which, the determinant factorizes.

For example, if propagation is along any of the three crystal axes $\left(x_1, x_2, x_3\right)$ Eq. (2.33) reduces to the following

$$\left(p^2 C_{11} - \rho\right)\left(p^2 C_{44} - \rho\right)\left(p^2 C_{44} - \rho\right) = 0 \quad . \qquad (2.34)$$

This gives three solutions of p^2 as

$$\begin{aligned} &(1)\ p^2 = \frac{\rho}{C_{11}}, \\ &(2)\ p^2 = \frac{\rho}{C_{44}}, \\ &(3)\ p^2 = \frac{\rho}{C_{44}}. \end{aligned} \qquad (2.35)$$

Assuming propagation along x1 axis for the moment Christoffel equation (2.32) reduces to

$$\boxed{\left(l_1 = 1, l_2 = l_3 = 0\right)}$$

$$p^2 \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{44} & 0 \\ 0 & 0 & C_{44} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \rho \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad . \tag{2.36}$$

The above equation shows that for propagation along $x1$ axis, the motion is split into pure P, SV and SH modes. P motion is along the direction of propagation $x1$ with speed $\sqrt{\frac{C_{11}}{\rho}}$, SH motion is along the $x2$ axis with speed $\sqrt{\frac{C_{44}}{\rho}}$, and the SV motion is along the x_3 axis with speed $\sqrt{\frac{C_{44}}{\rho}}$. Thus P is pure longitudinal and SH and SV are pure shears.

Next we consider propagation in x_1x_3 plane such that $l_2 = 0$,

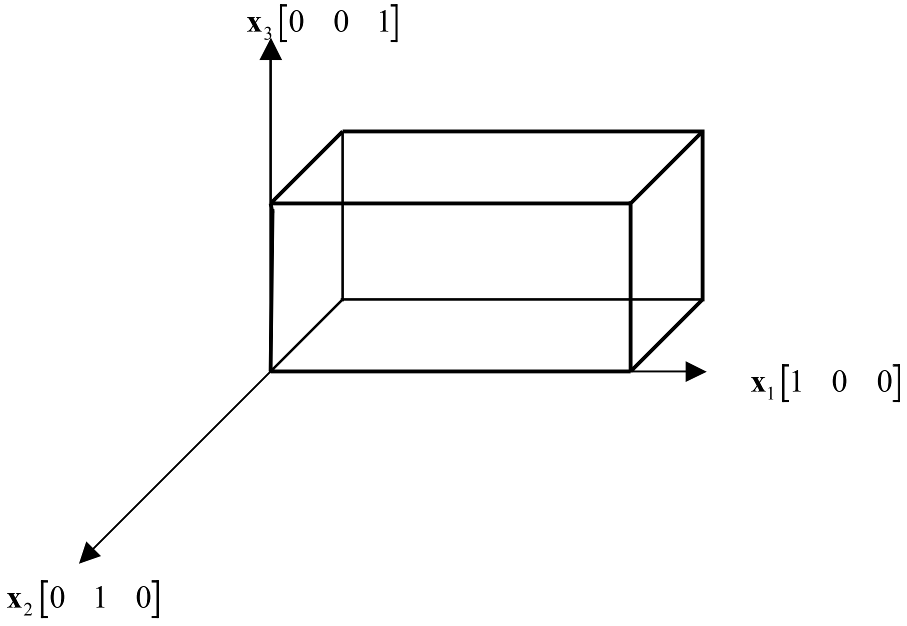

Figure 3.1.

For this, the Christoffel equation (2.32) reduces to

$$\begin{pmatrix} p^2\left(C_{11}l_1^2 + C_{44}l_3^2\right) - \rho & 0 & p^2\left(C_{12} + C_{44}\right)l_1 l_3 \\ 0 & p^2 C_{44} - \rho & 0 \\ p^2\left(C_{12} + C_{44}\right)l_1 l_3 & 0 & p^2\left(C_{11}l_1^2 + C_{44}l_3^2\right) - \rho \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0, \tag{2.37}$$

and the characteristic equation takes the following form

$$\begin{vmatrix} p^2\left(C_{11}l_1^2 + C_{44}l_3^2\right) - \rho & 0 & p^2\left(C_{12} + C_{44}\right)l_1 l_3 \\ 0 & p^2 C_{44} - \rho & 0 \\ p^2\left(C_{12} + C_{44}\right)l_1 l_3 & 0 & p^2\left(C_{11}l_3^2 + C_{44}l_1^2\right) - \rho \end{vmatrix} = 0 \tag{2.38}$$

or,

$$\left(p^2 C_{44} - \rho\right)\left[\left\{p^2\left(C_{11}l_1^2 + C_{44}l_3^2\right) - \rho\right\}\left\{p^2\left(C_{11}l_3^2 + C_{44}l_1^2\right) - \rho\right\} - p^4\left(C_{11} + C_{44}\right)^2 l_1^2 l_3^2\right] = 0. \tag{2.39}$$

Thus the Christoffel equation decouples into a linear equation

$$\left(p^2 C_{44} - \rho\right) = 0, \tag{2.40}$$

and quadratic form

$$\left[\left\{p^2\left(C_{11}l_1^2 + C_{44}l_3^2\right) - \rho\right\}\left\{p^2\left(C_{11}l_3^2 + C_{44}l_1^2\right) - \rho\right\} - p^4\left(C_{11} + C_{44}\right)^2 l_1^2 l_3^2\right] = 0. \tag{2.41}$$

It is straightforward to find roots of equation (2.40). To find roots of equation (2.41), we need to do some algebra and we can show that the three eigen values for propagation in x_1x_3 plane are given by

$$
\begin{aligned}
&(1)\ p_1 = \sqrt{\frac{\rho}{C_{44}}}, \\
&(2)\ p_2 = (2\rho)^{\frac{1}{2}}\left\{C_{11} + C_{44} - \sqrt{(C_{11} - C_{44})^2 \cos^2 2\theta + (C_{12} + C_{44})^2 \sin^2 2\theta}\right\}^{-\frac{1}{2}}, \\
&(3)\ p_3 = (2\rho)^{\frac{1}{2}}\left\{C_{11} + C_{44} - \sqrt{(C_{11} - C_{44})^2 \cos^2 2\theta + (C_{12} + C_{44})^2 \sin^2 2\theta}\right\}^{-\frac{1}{2}},
\end{aligned}
\tag{2.42}
$$

where $l_3 = \cos\theta$ and $l_1 = \sin\theta$. Substituting the first of Eq. (2.42) into the Christoffel equation, we immediately see that this solution is along x_2 direction only and therefore, perpendicular to the plane of propagation and hence is a pure shear (SH) mode. Solutions (2) and (3) in Eq. (2.42) are *quasi-longitudinal* waves respectively. However, they do reduce to pure modes for special propagation directions

Case I: $\theta = 0$, then $l_1 = 0$ and $l_3 = 1, \widehat{l} = \widehat{x}_3$. From Eq. (2.42) we have

$$
\begin{aligned}
p_2 &= (2\rho)^{\frac{1}{2}}\{C_{11} + C_{44} - C_{11} + C_{44}\}^{-\frac{1}{2}} = \sqrt{\frac{\rho}{C_{44}}}, \\
p_3 &= (2\rho)^{\frac{1}{2}}\{C_{11} + C_{44} + C_{11} - C_{44}\}^{-\frac{1}{2}} = \sqrt{\frac{\rho}{C_{11}}}.
\end{aligned}
\tag{2.43}
$$

The Christoffel equation becomes

$$
\begin{pmatrix}
p^2C_{44} - \rho & 0 & 0 \\
0 & p^2C_{44} - \rho & 0 \\
0 & 0 & p^2C_{11} - \rho
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0. \tag{2.44}
$$

Substituting p_2 or p_1 into Eq. (2.44) yields two pure shear waves, one with motion along x_1 axis and the other along x_2. Substituting p_3 for p, yields motion along x_3 and hence it is pure longitudinal.

Therefore at $\theta = 0$, we get two pure shear and one pure longitudinal motion.

Case II: $\theta = \frac{\pi}{2}$, then $l_1 = 1$ and $l_3 = 0, \hat{l} = \hat{x}_1$. From Eq. (2.42) we have

$$\begin{aligned} p_1 &= \sqrt{\frac{\rho}{C_{44}}}, \\ p_2 &= (2\rho)^{\frac{1}{2}}\{C_{11} + C_{44} - C_{11} + C_{44}\}^{-\frac{1}{2}} = \sqrt{\frac{\rho}{C_{44}}}, \\ p_3 &= (2\rho)^{\frac{1}{2}}\{C_{11} + C_{44} + C_{11} - C_{44}\}^{-\frac{1}{2}} = \sqrt{\frac{\rho}{C_{11}}}. \end{aligned} \tag{2.45}$$

Substituting these into the Christoffel equation, we obtain two pure shear motions along x_2 and x_3 and one pure longitudinal motion along x_1 direction.

Thus at $\theta = \frac{\pi}{2}$, we again obtain two pure shear and one pure longitudinal motions with slownesses

$$\begin{aligned} p_1 &= \sqrt{\frac{\rho}{C_{44}}}, \\ p_2 &= \sqrt{\frac{\rho}{C_{44}}}, \\ p_3 &= \sqrt{\frac{\rho}{C_{11}}}. \end{aligned}$$

Case III: Besides the above two angles, at $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ also, the quasi-shear and quasi-longitudinal motions become pure shear and longitudinal respectively. From the first row of Christoffel equation (2.37), we have

$$\left[p^2\left(C_{11}l_1^2 + C_{44}l_3^2\right) - \rho\right]u_1 + p^2\left(C_{12} + C_{44}\right)l_1 l_3 u_3 = 0,$$

or,

$$\frac{u_1}{u_3} = \frac{p^2\left(C_{12} + C_{44}\right)l_1 l_3}{p^2\left(C_{11}l_1^2 + C_{44}l_3^2\right) - \rho} \quad . \tag{2.46}$$

At $\theta = \dfrac{\pi}{4}, l_1 = l_3 = \dfrac{1}{\sqrt{2}}$, therefore we have

$$\frac{u_1}{u_3} = -\frac{p^2\left(C_{12} + C_{44}\right)/2}{p^2\left(C_{11} + C_{44}\right)/2 - \rho} \quad . \tag{2.47}$$

And at $\theta = \dfrac{3\pi}{4}, -\dfrac{1}{\sqrt{2}}, l_3 = \dfrac{1}{\sqrt{2}}$, therefore we have

$$\frac{u_1}{u_3} = -\frac{p^2\left(C_{12} + C_{44}\right)/2}{p^2\left(C_{11} + C_{44}\right)/2 - \rho} \quad . \tag{2.48}$$

Again at $\theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{3\pi}{4}$, $\cos^2 2\theta = 0$, and $\sin^2 2\theta = 1$. Thus from Eq. (2.42), we have

$$\begin{aligned} p_2^2 &= \frac{2\rho}{\left[\left(C_{11} + C_{44}\right) - \left(C_{12} + C_{44}\right)\right]}, \\ p_2^2\left(\frac{C_{11} + C_{44}}{2}\right) - \rho &= -p_3^2\left(\frac{C_{12} + C_{44}}{2}\right). \end{aligned} \tag{2.49}$$

Similarly, we obtain

$$p_3^2\left(\frac{C_{11} + C_{44}}{2}\right) - \rho = -p_3^2\left(\frac{C_{12} + C_{44}}{2}\right). \tag{2.50}$$

From Eqs. (2.47), (2.48), (2.49), and (2.50) we obtain the following equations

$$\begin{aligned}
&\frac{u_1}{u_3} = -1 \; for\, solution\, 2\, at\, \theta = \frac{\pi}{4}, \\
&\frac{u_1}{u_3} = 1 \; for\, solution\, 3\, at\, \theta = \frac{\pi}{4}, \\
&\frac{u_1}{u_3} = 1 \; for\, solution\, 2\, at\, \theta = \frac{3\pi}{4}, \\
&\frac{u_1}{u_3} = -1 \; for\, solution\, 3\, at\, \theta = \frac{3\pi}{4}.
\end{aligned} \tag{2.51}$$

Since u_2 is identically zero for all these solutions, at $\theta = \frac{\pi}{4}$, the motion for the solution takes the following form

$$\mathbf{u} = u_1(\hat{x}_1 - \hat{x}_3) \tag{2.52}$$

The propagation direction at $\theta = \frac{\pi}{4}$ is given by

$$\hat{l} = \frac{1}{\sqrt{2}}\hat{x}_1 + \frac{1}{\sqrt{2}}\hat{x}_3 \quad . \tag{2.53}$$

From Eqs. (2.52) and (2.53) we have

$$\hat{l}.\mathbf{u} = \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_1 = 0 \quad . \tag{2.54}$$

Thus the solution (2) is pure shear at $\theta = \frac{\pi}{4}$.

Again at this angle it follows from Eq. (2.51) that the motion **u** for solution 3 is of the form

$$\mathbf{u} = u_1\left(\hat{x}_1 + \hat{x}_3\right) \quad . \tag{2.55}$$

Thus we have

$$\hat{l}X\mathbf{u} = \frac{1}{\sqrt{2}}\left[\left(\hat{x}_1 + \hat{x}_3\right)X\left(\hat{x}_1 + \hat{x}_3\right)\right] = \mathbf{0} \quad . \tag{2.56}$$

Thus the solution (3) is pure longitudinal at $\theta = \frac{\pi}{4}$.

Similarly noting that at $\theta = \frac{3\pi}{4}$, the direction of propagation is given by:

$$l = -\frac{1}{\sqrt{2}}\hat{x}_1 + \frac{1}{\sqrt{2}}\hat{x}_3, \tag{2.57}$$

and from the forms of displacement vector **u** at $\theta = \frac{3\pi}{4}$ (Eq. 2.51), we find that solutions 2 and 3 are pure shear and longitudinal respectively at this angle of incidence as well.

Also from Eqs. (2.49) and (2.50), we find for $\theta = \frac{\pi}{4}$, and $\theta = \frac{3\pi}{4}$

$$\begin{aligned} p_1 &= \sqrt{\frac{2\rho}{C_{11} - C_{12}}}, \\ p_2 &= \sqrt{\frac{2\rho}{C_{11} + C_{12} + 2C_{44}}}. \end{aligned} \tag{2.58}$$

Thus, to summarize, we find the following properties for the propagation of plane waves in a cubic srystal:

- For propagation along any of the crystal axes, the wave motion decouples into one pure longitudinal and two pure shear motions; longitudinal motion being along the axis of propagation and shear motion being along the other two axes orthogonal to it. The slowness of these three motions are given by

$$p_1 = \sqrt{\frac{\rho}{C_{44}}}, \quad \text{pure shear}$$

$$p_2 = \sqrt{\frac{\rho}{C_{44}}}, \quad \text{pure shear}$$

$$p_3 = \sqrt{\frac{\rho}{C_{11}}}, \quad \text{pure longitudinal.}$$

•For propagation direction along any of the coordinate planes, the motion decouples into one pure shear motion polarized along an axis normal to the plane of propagation and a quasi-longitudinal and quasi-shear system. The quasi-longitudinal and quasi-shear, however, become pure longitudinal and pure shear respectively, at the angles of incidence 0, $\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$, and $\theta = \frac{3\pi}{4}$. Thus the three solutions at these angles become:

At $\theta = 0$ and $\theta = \frac{\pi}{2}$

$$p_1 = \sqrt{\frac{\rho}{C_{44}}}, \quad \text{pure shear}$$

$$p_2 = \sqrt{\frac{\rho}{C_{44}}}, \quad \text{pure shear}$$

$$p_3 = \sqrt{\frac{\rho}{C_{11}}}, \quad \text{pure longitudinal.}$$

At $\theta = \frac{\pi}{4}$, and $\theta = \frac{3\pi}{4}$

$$p_1 = \sqrt{\frac{\rho}{C_{44}}}, \quad \text{pure shear}$$

$$p_2 = \sqrt{\frac{2\rho}{C_{11} - C_{22}}}, \quad \text{pure shear}$$

$$p_3 = \sqrt{\frac{2\rho}{C_{11} + C_{12} + 2C_{44}}}, \quad \text{pure longitudinal.}$$

At any other angle slownesses are given by Eq. (2.42).

Gallium Arsenide (GaAs) with the following material properties:

Symmetry class: Cubic

Density $(r) = 5307\, kg / m^3$,

$C_{11} = 11.88 X 10^{10}\, newton / m^2$,

$C_{44} = 5.94 X 10^{10}\, newton / m^2$,

$C_{12} = 15.38 X 10^{10}\, newton / m^2$.