

## LECTURE 8

### *Hexagonal Symmetry: Transverse Isotropy*

The elastic coefficient matrix for a hexagonally symmetric medium where  $x_3$  is the axis of symmetry, is given by

$$C = \begin{pmatrix} C_{11} & C_{11} - 2C_{66} & C_{13} & 0 & 0 & 0 \\ C_{11} - 2C_{66} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}.$$

The Christoffel equation for such a medium is given by

$$\begin{pmatrix} p^2(C_{11}l_1^2 + C_{66}l_2^2 + C_{44}l_3^2) - \rho & p^2(C_{12} + C_{66})l_1l_2 & p^2(C_{13} + C_{44})l_1l_3 \\ p^2(C_{12} + C_{66})l_1l_2 & p^2(C_{66}l_1^2 + C_{11}l_2^2 + C_{44}l_3^2) - \rho & p^2(C_{13} + C_{44})l_2l_3 \\ p^2(C_{13} + C_{44})l_1l_3 & p^2(C_{13} + C_{44})l_2l_3 & p^2(C_{44}l_1^2 + C_{44}l_2^2 + C_{33}l_3^2) - \rho \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \mathbf{0}.$$

where

$$C_{12} = C_{11} - 2C_{66}$$

This gives the characteristic equation for the hexagonal symmetry as

$$\begin{vmatrix} p^2(C_{11}l_1^2 + C_{66}l_2^2 + C_{44}l_3^2) - \rho & p^2(C_{12} + C_{66})l_1l_2 & p^2(C_{13} + C_{44})l_1l_3 \\ p^2(C_{12} + C_{66})l_1l_2 & p^2(C_{66}l_1^2 + C_{11}l_2^2 + C_{44}l_3^2) - \rho & p^2(C_{13} + C_{44})l_2l_3 \\ p^2(C_{13} + C_{44})l_1l_3 & p^2(C_{13} + C_{44})l_2l_3 & p^2(C_{44}l_1^2 + C_{44}l_2^2 + C_{33}l_3^2) - \rho \end{vmatrix} = 0.$$

For propagation in  $x_1x_2$  plane, we set  $l_3=0$  to obtain the following Chritoffel equaion

$$\begin{pmatrix} p^2(C_{11}l_1^2 + C_{66}l_2^2) - \rho & p^2(C_{12} + C_{66})l_1l_2 & 0 \\ p^2(C_{12} + C_{66})l_1l_2 & p^2(C_{66}l_1^2 + C_{11}l_2^2) - \rho & 0 \\ 0 & 0 & p^2(C_{44}l_1^2 + C_{44}l_2^2) - \rho \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \mathbf{0}.$$

and the characteristic equation

$$\begin{vmatrix} p^2(C_{11}l_1^2 + C_{66}l_2^2) - \rho & p^2(C_{12} + C_{66})l_1l_2 & 0 \\ p^2(C_{12} + C_{66})l_1l_2 & p^2(C_{66}l_1^2 + C_{11}l_2^2) - \rho & 0 \\ 0 & 0 & p^2(C_{44}l_1^2 + C_{44}l_2^2) - \rho \end{vmatrix} = \mathbf{0}.$$

Thus for propagation in the  $x_1x_2$  plane in a hexagonal crystal, with  $x_3$  as the axis of symmetry, the wave motion decouples into a linear part

$$p^2 C_{44} - \rho = 0 \quad ,$$

or,

$$p_1 = \sqrt{\frac{\rho}{C_{44}}} \quad . \quad )$$

It is clear from the above equation that it corresponds to particle the z direction or normal to the plane of propagation (SH mode).

The determinant equation can be expanded as.

$$\begin{aligned} & (p^2(C_{11}l_1^2 + C_{66}l_2^2) - \rho)(p^2(C_{66}l_1^2 + C_{11}l_2^2) - \rho)(p^2(C_{44}l_1^2 + C_{44}l_2^2) - \rho) - \\ & (p^2(C_{12} + C_{66})l_1l_2)(p^2(C_{12} + C_{66})l_1l_2)(p^2(C_{44}l_1^2 + C_{44}l_2^2) - \rho) = \mathbf{0}. \end{aligned}$$

or

$$\begin{aligned} & (p^2(C_{11}l_1^2 + C_{66}l_2^2) - \rho)(p^2(C_{66}l_1^2 + C_{11}l_2^2) - \rho)(p^2(C_{44}) - \rho) - \\ & (p^2(C_{12} + C_{66})l_1l_2)(p^2(C_{12} + C_{66})l_1l_2)(p^2(C_{44}) - \rho) = \mathbf{0}. \end{aligned}$$

Therefore the quadratic part is give by

$$(p^2(C_{11}l_1^2 + C_{66}l_2^2) - \rho)(p^2(C_{66}l_1^2 + C_{11}l_2^2) - \rho) - (p^2(C_{12} + C_{66})l_1l_2)(p^2(C_{12} + C_{66})l_1l_2) = \mathbf{0}.$$

which reduces to the following equation

$$(p^2C_{66} - \rho)(p^2C_{11} - \rho) = 0 \quad .$$

Thus the quadratic part factorizes into the following

$$p_2 = \sqrt{\frac{\rho}{C_{66}}} \quad ,$$

and,

$$p_3 = \sqrt{\frac{\rho}{C_{11}}} \quad .$$

To find the direction of polarization for these solutions, we find from the first row of the matirx:

$$\frac{u_1}{u_2} = -\frac{p^2(C_{11} - C_{66})l_1l_2}{[p^2(C_{11}l_1^2 + C_{66}l_2^2) - \rho]} \quad .$$

For solution 2, with  $p = \sqrt{\frac{\rho}{C_{66}}}$ , we have

$$\frac{u_1}{u_2} = -\frac{\frac{\rho}{C_{66}}(C_{11} - C_{66})l_1l_2}{\left[\frac{\rho}{C_{66}}(C_{11}l_1^2 + C_{66}l_2^2) - \rho\right]} = -\frac{(C_{11} - C_{66})l_1l_2}{\left[(C_{11}l_1^2 + C_{66}l_2^2) - C_{66}\right]} \quad .$$

Similarly for solution 3, with  $p = \sqrt{\frac{\rho}{C_{11}}}$ , we have

$$\frac{u_1}{u_2} = -\frac{(C_{11} - C_{66})l_1l_2}{[(C_{11}l_1^2 + C_{66}l_2^2) - C_{11}]} .$$

For waves propagating in  $x_1x_2$  plane,  $\hat{l}$  along the direction of propagation is given by

$$\hat{l} = l_1\hat{x}_1 + l_2\hat{x}_2 .$$

From Eq. (2.67) the particle motion  $\mathbf{u}$  for solution 2 is proportional to

$$\mathbf{u} = -\frac{(C_{11} - C_{66})l_1l_2}{C_{11}l_1^2 + C_{66}l_2^2 - C_{66}}\hat{x}_1 + \hat{x}_2 .$$

Thus we have

$$\begin{aligned} \hat{l} \cdot \mathbf{u} &= (l_1\hat{x}_1 + l_2\hat{x}_2) \cdot \left[ -\frac{(C_{11} - C_{66})l_1l_2}{(C_{11}l_1^2 + C_{66}l_2^2 - C_{66})}\hat{x}_1 + \hat{x}_2 \right] \\ &= l_2 - \frac{(C_{11} - C_{66})l_1^2l_2}{(C_{11}l_1^2 + C_{66}l_2^2 - C_{66})} = 0 \end{aligned}$$

Thus solution 2 is pure shear polarized in  $x_1x_2$  plane (the plane of propagation).

Similarly we can show that solution 3 is pure longitudinal.

Thus for propagation in  $x_1x_2$  plane in a hexagonal crystal with  $x_3$  as the axis of symmetry, the motion polarizes in three orthogonal directions:

- $p_1 = \sqrt{\frac{\rho}{C_{44}}}$ , Corresponds to pure shear motion along  $x_3$  axis (normal to the plane of propagation – a pure SH mode.

- $p_2 = \sqrt{\frac{\rho}{C_{66}}}$ , Corresponds to pure shear motion in the plane of propagation – a pure SV mode.

And,

- $p_3 = \sqrt{\frac{\rho}{C_{11}}}$ , Corresponds to pure longitudinal motion – pure P wave.

Thus in the  $x_1x_2$  plane three pure modes exist and they propagate with different velocities. Note also that there is no azimuthal variation of the above three wave speeds; so the slowness curves are circle like an isotropic medium. This is the reason to call a hexagonally symmetric medium with  $x_3$  (the vertical) as the axis of symmetry, *transversely isotropic* medium.