REFLECTIVITY 4

Propagators and Wave Propagators for Our Seismic System

We will work with our system of ODE $\partial_z \boldsymbol{b} = i\omega \boldsymbol{A}\boldsymbol{b} + \boldsymbol{f}$ for a general stratified anisotropic medium.

<u>Eigenvectors and Eigenvalues</u>

We denote the eigenvector matrix of $\boldsymbol{A}$ by $\boldsymbol{D}$ i.e., each column of $\boldsymbol{D}$ is an eigenvector of $\boldsymbol{A}$. Thus $\boldsymbol{AD}$=$\boldsymbol{D\Lambda}$ where $\boldsymbol{\Lambda} = diag(\Lambda_1, \Lambda_2, \ldots \Lambda_6)$. It can be shown that $\boldsymbol{\Lambda}_i s$ are the vertical slownesses.

Why is $\boldsymbol{D}$ useful?

To see why, first define a vector $\boldsymbol{v}$ such that $\boldsymbol{v} = \boldsymbol{D}^{-1}\boldsymbol{b}$ by left multiplication of the displacement stress vector $\boldsymbol{b}$ with the inverse of the eigenvector matrix and call it a wave vector for reasons which will be made clear later.

Recall

$$\partial_z \boldsymbol{b} = i\omega \boldsymbol{A}\boldsymbol{b}$$
$$\partial_z (\boldsymbol{D}\boldsymbol{v}) = i\omega \boldsymbol{D}\Lambda\boldsymbol{D}^{-1}\boldsymbol{D}\boldsymbol{v}$$
$$\boldsymbol{D}^{-1}\partial_z (\boldsymbol{D}\boldsymbol{v}) = i\omega \boldsymbol{\Lambda}\boldsymbol{v}$$
$$\boldsymbol{D}^{-1}(\partial_z \boldsymbol{D})\boldsymbol{v} + \partial_z \boldsymbol{v} = i\omega \boldsymbol{\Lambda}\boldsymbol{v}$$

In a homogeneous region $\partial_z \boldsymbol{D} = 0$ and we have $\partial_z \boldsymbol{v} = i\omega \boldsymbol{\Lambda}\boldsymbol{v}$ whose solution is given by

$$\boldsymbol{v}(z) = e^{i\omega \boldsymbol{\Lambda}(z - z_0)}\boldsymbol{v}(z_0) \qquad (19)$$

or, $\boldsymbol{v}(z) = \boldsymbol{Q}(z, z_0)\boldsymbol{v}(z_0)$, where $\boldsymbol{Q}(z, z_0) = e^{i\omega \boldsymbol{\Lambda}(z - z_0)}$ is called the wave propagator since it propagates the wave vector from z to z_0.

Thus we notice that each entry in $\boldsymbol{v}$ is proportional to the amplitude of an upgoing or downgoing plane wave (depending on the sign of the exponent) component and hence the name *wave vector* for $\boldsymbol{v}$.

Now, since

$$\boldsymbol{b}(z) = \boldsymbol{D}\boldsymbol{v}(z)$$
$$\boldsymbol{D}^{-1}\boldsymbol{b}(z) = \boldsymbol{v}(z) = e^{i\omega\Lambda(z-z_0)}\boldsymbol{v}(z_0)$$
$$\boldsymbol{b}(z) = \boldsymbol{D}e^{i\omega\Lambda(z-z_0)}\boldsymbol{D}^{-1}\boldsymbol{b}(z_0)$$
$$\boldsymbol{b}(z) = \boldsymbol{P}(z,z_0)\boldsymbol{b}(z_0)$$

where

$$\boxed{\boldsymbol{P}(z,z_0) = \boldsymbol{D}e^{i\omega\Lambda(z-z_0)}\boldsymbol{D}^{-1}} \tag{20}$$

Thus in a homogeneous region, the displacement-stress vector $\boldsymbol{b}$ can be propagated from a depth z_0 to a new depth z by using the propagator matrix $\boldsymbol{P}(z,z_0)$. Note that $\boldsymbol{P}(z_0,z_0) = \boldsymbol{I}$.

How do we get a formula for $\boldsymbol{P}(z,z_0)$ over an inhomogeneous region? Divide our model up into thin homogeneous layers and use the product rule for propagators

$$\boldsymbol{P}(z,z_0) = \boldsymbol{P}(z,z_N)\boldsymbol{P}(z_N,z_{N-1})\ldots\boldsymbol{P}(z_1,z_0) \tag{21}$$

Wave vectors are useful for applying radiation type boundary conditions. Since the components of the wave vector $\boldsymbol{v}$ are the coefficients of solutions that propagate with different (in general) vertical slownesses we are justified in naming the components of $\boldsymbol{v}$. For example we may call the first component of the wave vector the "*P*-wave amplitude". Recall that in a homogeneous region we have for fixed p_x, p_y, ω

$$\boldsymbol{v}(x,y,t,z) = e^{i\omega\left(p_x x + p_y y + \Lambda_1(z-z_0) - t\right)}\boldsymbol{v}(x,y,b,z)$$

This is a plane wave propagating with plane wave velocity C_1 given by

$$C_1^{-1} = \left(p_x^2 + p_y^2 + \Lambda_1^2\right)^{1/2}.$$

Thus $\Lambda_1 = \sqrt{1/C_1^2 - p_x^2 - p_y^2}$ gives the vertical slowness.

We will use the following notations as used in Fryer and Frazer (1984).

$$\begin{array}{lll} \Lambda_1 = q_P^u & \Lambda_2 = q_{S_1}^u & \Lambda_3 = q_{S_2}^u \\ \Lambda 4 = q_P^d & \Lambda_5 = q_{S_2}^d & \Lambda_6 = q_{S_3}^d \end{array} \tag{22}$$

Thus we shall assume that the 1st column of $\boldsymbol{D}$ is the eigenvector associated with upgoing *P*-wave, the 2nd column is the eigenvector of the upgoing $\boldsymbol{S_1}$ wave etc. That is, after we obtain $\boldsymbol{D}$ and $\boldsymbol{\Lambda}$ from $\boldsymbol{A}$ (by SVD or whatever) we rearrange terms in $\boldsymbol{\Lambda}$ and $\boldsymbol{D}$. It is fairly straightforward in isotropic media since up and downgoing can be identified from the sign of the eigenvalue and $v_p > v_s$. The eigenvalues for the two *S*-waves are identical for isotropic media (*SV* and *SH*). But how do we do this for anisotropic media?

Given our choice of FT and depth of $\boldsymbol{A}$ if follows from the radiation condition that

$$I_m\left(q^D\right) > 0 \text{ and } I_m\left(q^U\right) < 0 .$$

As in the isotropic case the elements of $\boldsymbol{v}$ may be identified with the amplitudes of upward and downward traveling plane waves.

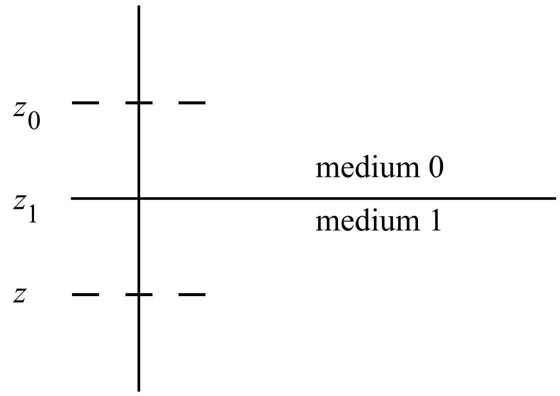
$$\boldsymbol{v} = \left[\boldsymbol{v}_U, v_D\right]^T = \left[\phi_U, \varphi_U, X_U, \phi_D, \varphi_D, X_D\right]^T \tag{23}$$

<u>Summary so far</u>: The propagator matrix is needed to propagate the displacement-stress vector $\boldsymbol{b}$ from one depth level to another. Recall that this requires that we find the eigenvalues (vertical slownesses), the eigenvector matrix $\boldsymbol{D}$ and its inverse $\boldsymbol{D}^{-1}$. In the isotropic case, these are known analytically, so the construction of the propagator is straight-forward. In the anisotropic case, analytic solutions can only be found for some simple symmetries. So, in general, solutions will be found numerically. Fortunately, $\boldsymbol{D}$ and its inverse are very simply related, as we shall show.

<u>Wave propagators in an inhomogeneous region</u>

We showed above that $\boldsymbol{v}(z) = e^{i\omega\boldsymbol{\Lambda}(z-z_0)}\boldsymbol{v}(z_0)$ (equation 19). Thus $e^{i\omega\boldsymbol{\Lambda}(z_0)}$ is the matrix that propagates the wave vector from z_0 to z and so we refer to it as the wave propagator.

In an inhomogeneous medium the propagator will not have such a simple form. For example, suppose that the medium has a discontinuity at z_1, we can derive the wave propagator from z_0 to z as follows.



Recall that $\boldsymbol{b}(z) = \boldsymbol{P}(z,z_0)\boldsymbol{b}(z_0)$ and $\boldsymbol{b} = \boldsymbol{D}\boldsymbol{v}$

$$
\begin{aligned}
\mathbf{D}_1^{-1}\mathbf{b}(z) &= \mathbf{D}_1^{-1}\mathbf{P}(z,z_1)\mathbf{P}(z_1,z_0)\mathbf{b}(z_0)\\
\mathrm{v}(z) &= \mathrm{D}_1^{-1}\mathrm{P}(z,z_1)\mathrm{P}(z_1,z_0)\mathrm{b}(z_0) = \mathrm{D}_1^{-1}\mathrm{D}_1\boldsymbol{e}^{\boldsymbol{i\omega}\mathrm{L}(z-z_1)}\mathrm{D}_1^{-1}\mathrm{D}_0\boldsymbol{e}^{\boldsymbol{i\omega}\mathrm{L}(z_1-z_0)}\mathrm{D}_0^{-1}\mathrm{b}(z_0)\\
\mathrm{v}(z) &= \boldsymbol{e}^{\boldsymbol{i\omega}\mathrm{L}(z-z_1)}\mathrm{D}_1^{-1}\mathrm{D}_0\boldsymbol{e}^{\boldsymbol{i\omega}\mathrm{L}(z_1-z_0)}\mathrm{v}(z_0)\\
\mathbf{v}(z) &= \mathbf{Q}(z,z_0)\mathbf{v}(z_0)
\end{aligned}
$$

Therefore

$$\boldsymbol{Q}(z,z_0) = e^{i\omega\boldsymbol{\Lambda}(z-z_1)}\boldsymbol{D}_1^{-1}\boldsymbol{D}_0 e^{i\omega\boldsymbol{\Lambda}(z_1-z_0)}$$

Wave propagators are usually called $\boldsymbol{Q}$ just as propagators are usually called $\boldsymbol{P}$. Note that we can write the propagators in terms of the wave propagators and vice versa.

$$
\begin{aligned}
\boldsymbol{b}(z) &= \boldsymbol{P}(z,z_0)\boldsymbol{b}(z_0)\\
\boldsymbol{v}(z) &= \boldsymbol{Q}(z,z_0)\boldsymbol{v}(z_0)
\end{aligned}
$$

and,

$$\boldsymbol{b}(z) = \boldsymbol{D}(z)\boldsymbol{v}(z) \text{ and } \boldsymbol{b}(z_0) = \boldsymbol{D}(z_0)\boldsymbol{v}(z_0)$$

or,

$$
\begin{aligned}
\boldsymbol{b}(z) &= \boldsymbol{D}(z)\boldsymbol{Q}(z,z_0)\boldsymbol{v}(z_0)\\
&= \boldsymbol{D}(z)\boldsymbol{Q}(z,z_0)\boldsymbol{D}^{-1}(z_0)\boldsymbol{b}(z_0)
\end{aligned}
$$

Thus,

$$P(z,z_0) = D(z)Q(z,z_0)D^{-1}(z_0)$$
$$Q(z,z_0) = D^{-1}(z)P(z,z_0)D(z_0)$$

We know that

$$P(z_2,z_0) = P(z_2,z_1)P(z_1,z_0).$$

Therefore

$$\begin{aligned} D(z_2)Q(z_2,z_0)D_0^{-1}(z_0) &= D(z_2)Q(z_2,z_1)D_1^{-1}(z_0)D_1Q(z_1,z_0)D_0^{-1}(z_0) \\ &= D(z_2)Q(z_2,z_1)Q(z_1,z_0)D_0^{-1}(z_0) \end{aligned}$$

or,

$$Q(z_2,z_0) = Q(z_2,z_1)Q(z_1,z_0).$$

Where z_i's are the depths to the discontinuities. It is important to note that ***P*** is continuous at interfaces in the medium but ***Q*** is not. That is, $Q(z_i^+, z_i^-)$ is not an indentity matrix.