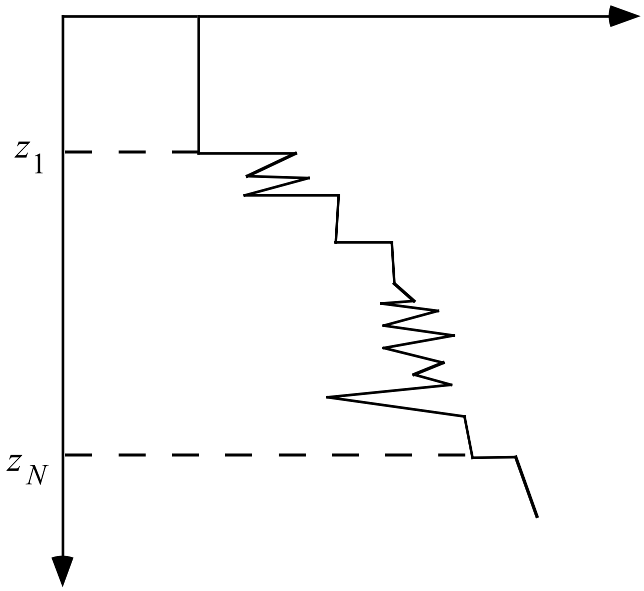
REFLECTIVITY 5

Reflection and Transmission

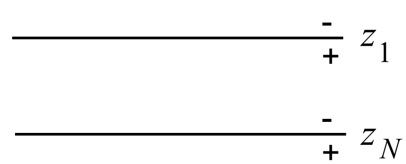


Suppose that we have an earth model homogeneous above z_1 and below z_N and inhomogeneous between.

Then it is reasonable to talk about the reflection and transmission properties of such a zone and to define R/T coefficients.

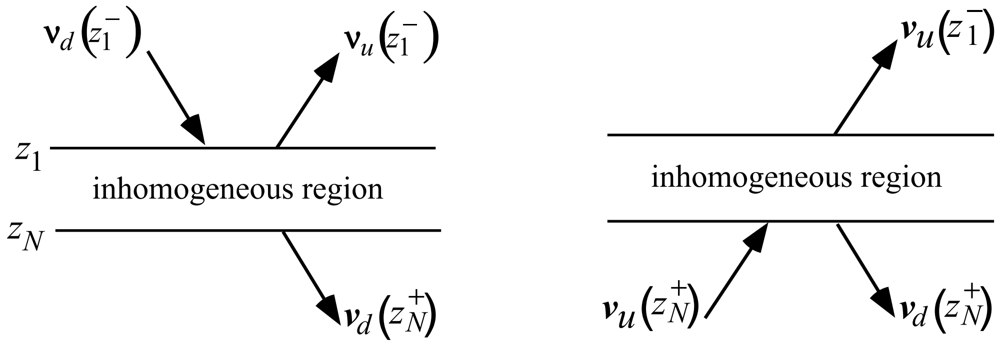
We have $\boldsymbol{b}(z_N) = \boldsymbol{P}(z_N, z_1)\boldsymbol{b}(z_1)$ and $\boldsymbol{v}(z_N^+) = \boldsymbol{Q}(z_N^+, z_1^-)\boldsymbol{v}(z_1^-)$.

Notation:



where, as before, $\boldsymbol{P}$ is a propagator and $\boldsymbol{Q}$ is a wave propagator. We write $\boldsymbol{v} = (\boldsymbol{v}_u, \boldsymbol{v}_d)^T$; $\boldsymbol{v}_u = [\phi_u, \varphi_u, x_u]^T$ and $\boldsymbol{v}_d = [\phi_d, \varphi_d, x_d]^T$. This means that in the equation $\boldsymbol{b} = \boldsymbol{D}\boldsymbol{v}$ we are using a $\boldsymbol{D}$ such that its first column is an upgoing p-wave etc.

We now solve two reflection problems



I: Wave incident from above II: Wave incident from below

Case I:

$$\begin{pmatrix} 0 \\ \boldsymbol{v}_d(z_N^+) \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \hline \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{array}\right) \begin{pmatrix} \boldsymbol{v}_u(z_1^-) \\ \boldsymbol{v}_d(z_1^-) \end{pmatrix} \tag{34}$$

Set

$$\boldsymbol{Q}(z_N^+, z_1^-) = \left(\begin{array}{c|c} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \hline \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{array}\right)$$

which gives

$$0 = \boldsymbol{Q}_{11}\boldsymbol{v}_u(z_1^-) + \boldsymbol{Q}_{12}\boldsymbol{v}_d(z_1^-)$$
$$\boldsymbol{v}_d(z_N^+) = \boldsymbol{Q}_{21}\boldsymbol{v}_u(z_1^-) + \boldsymbol{Q}_{22}\boldsymbol{v}_d(z_1^-)$$

or,

$$\begin{aligned} \boldsymbol{v}_u(z_1^-) &= \left(-\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12}\right)\boldsymbol{v}_d(z_1^-) \\ \boldsymbol{v}_d(z_N^+) &= \left(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12}\right)\boldsymbol{v}_d(z_1^-) \end{aligned} \tag{35}$$

Thus we are justified in defining matrices of reflection and transmission coefficients

$$\boldsymbol{R}_D = -\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12};\ \boldsymbol{T}_D = \boldsymbol{Q}_{22} - \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12} \tag{36}$$

Case II:

$$\begin{pmatrix} \boldsymbol{v}_u\left(z_N^+\right) \\ \boldsymbol{v}_d\left(z_N^+\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_u\left(z_1^-\right) \\ 0 \end{pmatrix} \tag{37}$$

$$\boldsymbol{v}_u\left(z_N^+\right) = \boldsymbol{Q}_{11}\boldsymbol{v}_u\left(z_1^-\right)$$

$$\boldsymbol{v}_d\left(z_N^+\right) = \boldsymbol{Q}_{21}\boldsymbol{v}_u\left(z_1^-\right)$$

or,

$$\begin{aligned} \boldsymbol{v}_u\left(z_1^-\right) &= \boldsymbol{Q}_{11}^{-1}\boldsymbol{v}_u\left(z_N^+\right) \\ \boldsymbol{v}_d\left(z_N^+\right) &= \boldsymbol{Q}_{21}\boldsymbol{v}_u\left(z_1^-\right) = \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{v}_u\left(z_N^+\right) \end{aligned} \tag{38}$$

Thus we are justified in defining

$$\boldsymbol{T}_u = \boldsymbol{Q}_{11}^{-1};\ \boldsymbol{R}_u = \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1} \tag{39}$$

Since

$$\boldsymbol{v}_u\left(z_1^-\right) = \boldsymbol{R}_D\boldsymbol{v}_d\left(z_1^-\right)$$

It is clear that

$$\boldsymbol{R}_D = \begin{pmatrix} r_{pp}^D & r_{p1}^D & r_{p2}^S \\ r_{1p}^D & r_{11}^D & r_{12}^D \\ r_{2p}^D & r_{21}^D & r_{22}^D \end{pmatrix}$$

r_{pp}^D reflection coefficient for a downgoing *P*-wave to an upgoing *p*-wave and so on.

Similarly,

$$\boldsymbol{T}_D = \begin{pmatrix} t_{pp}^D & t_{p1}^D & t_{p2}^D \\ t_{1p}^D & t_{11}^D & t_{12}^D \\ t_{2p}^D & t_{21}^D & t_{22}^D \end{pmatrix} \quad \boldsymbol{T}_u = \begin{pmatrix} t_{pp}^u & t_{p1}^u & t_{p2}^u \\ t_{1p}^u & t_{11}^u & t_{12}^u \\ t_{2p}^u & t_{21}^u & t_{22}^u \end{pmatrix} \quad \boldsymbol{R}_u = \begin{pmatrix} r_{pp}^u & r_{p1}^u & r_{p2}^u \\ r_{1p}^u & r_{11}^u & r_{12}^u \\ r_{2p}^u & r_{21}^u & r_{22}^u \end{pmatrix}.$$

We now put all of these R/T coefficients in a big 6X6 matrix.

$$\boldsymbol{R} = \left(\begin{array}{c:c} \boldsymbol{R}_D & \boldsymbol{T}_u \\ \hdashline \boldsymbol{T}_D & \boldsymbol{R}_u \end{array}\right) = \left(\begin{array}{c:c} \boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12} & \boldsymbol{Q}_{11}^{-1} \\ \hdashline \boldsymbol{Q}_{22} - \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12} & \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1} \end{array}\right) \tag{40}$$

Remember that the $\boldsymbol{Q}$ in this equation is downward $\boldsymbol{Q}$, i.e. $\boldsymbol{Q} = \left(z_N^+, z_1^-\right)$.

From equations (36) and (39) we have

$$\boldsymbol{R}_D = -\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12} \tag{41}$$

$$\boldsymbol{T}_D = \boldsymbol{Q}_{22} - \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12} \tag{42}$$

$$\boldsymbol{T}_u = \boldsymbol{Q}_{11}^{-1} \tag{43}$$

$$\boldsymbol{R}_u = \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1} \tag{44}$$

From equation (43)

$$\boldsymbol{Q}_{11} = \boldsymbol{T}_U^{-1} \tag{45}$$

From equations (41) and (45)

$$\begin{aligned} \boldsymbol{Q}_{12} &= -\boldsymbol{Q}_{11}\boldsymbol{R}_D \\ \boldsymbol{Q}_{12} &= \boldsymbol{T}_u^{-1}\boldsymbol{R}_D \end{aligned} \tag{46}$$

From equation (44)

$$\boldsymbol{Q}_{21} = \boldsymbol{R}_u\boldsymbol{Q}_{11} = \boldsymbol{R}_u\boldsymbol{T}_u^{-1} \tag{47}$$

From equations (42) and (41)

$$\boldsymbol{T}_D = \boldsymbol{Q}_{22} + \boldsymbol{Q}_{21}\boldsymbol{R}_D$$

$$\boldsymbol{Q}_{22} = \boldsymbol{T}_D - \boldsymbol{R}_u\left(\boldsymbol{T}_u\right)^{-1}\boldsymbol{R}_D$$

Thus we have,

$$\boldsymbol{Q}\left(z_N^+, z_1^-\right) = \left[\begin{array}{c:c} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \hdashline \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{array}\right] = \begin{bmatrix} \boldsymbol{T}_u^{-1} & -\left(\boldsymbol{T}_u\right)^{-1}\boldsymbol{R}_D \\ \boldsymbol{R}_u\left(\boldsymbol{T}_u\right)^{-1} & \boldsymbol{T}_D - \boldsymbol{R}_u\left(\boldsymbol{T}_u\right)^{-1}\boldsymbol{R}_D \end{bmatrix} \tag{48}$$

Thus we have expressed the wave propagator Q in terms of the reflection coefficients.

We can now derive a similar expression for the upward propagator, $\boldsymbol{Q}\left(z_1^-, z_N^+\right)$. Thus for the first case instead of using equation (34) we use

$$\begin{pmatrix} \boldsymbol{v}_u\left(z_1^-\right) \\ \boldsymbol{v}_d\left(z_1^-\right) \end{pmatrix} = \boldsymbol{Q}^u\left(z_1^-, z_N^+\right)\begin{pmatrix} 0 \\ \boldsymbol{v}_d\left(z_N^+\right) \end{pmatrix} \tag{49}$$

and in the second case instead of using equation (37) we use

$$\begin{pmatrix} \boldsymbol{v}_u\left(z_N^+\right) \\ 0 \end{pmatrix} = \boldsymbol{Q}^u\left(z_1^-, z_N^+\right)\begin{pmatrix} \boldsymbol{v}_u\left(z_N^+\right) \\ \boldsymbol{v}_d\left(z_N^+\right) \end{pmatrix} \tag{50}$$

After doing similar algebra, we obtain

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_D & \boldsymbol{T}_u \\ \boldsymbol{T}_D & \boldsymbol{R}_u \end{pmatrix} = \left(\begin{array}{c:c} \boldsymbol{Q}_{12}^u \boldsymbol{Q}_{22}^{u-1} & \boldsymbol{Q}_{11}^u - \boldsymbol{Q}_{12}^u \boldsymbol{Q}_{22}^{u-1} \boldsymbol{Q}_{21} \\ \hdashline \boldsymbol{Q}_{22}^{u-1} & \boldsymbol{Q}_{22}^{u-1} \boldsymbol{Q}_{21}^u \end{array}\right)$$

and,

$$\boldsymbol{Q}\left(z_1^-, z_N^+\right) = \begin{bmatrix} \boldsymbol{Q}_{11}^u & \boldsymbol{Q}_{12}^u \\ \boldsymbol{Q}_{21}^u & \boldsymbol{Q}_{22}^u \end{bmatrix}$$

$$\boldsymbol{Q}\left(z_1^-, z_N^+\right) = \left[\begin{array}{c:c} \boldsymbol{T}_u - \boldsymbol{R}_D\left(\boldsymbol{T}_D\right)^{-1}\boldsymbol{R}_u & \boldsymbol{R}_D\left(\boldsymbol{T}_D\right)^{-1} \\ \hdashline -\left(\boldsymbol{T}_D\right)^{-1}\boldsymbol{R}_u & \left(\boldsymbol{T}_D\right)^{-1} \end{array}\right] \tag{51}$$

This is the most fundamental lemma of Kennett (1974) from which (almost) everything (that's any good) follows!

<u>Iteration Equation</u>

We now establish the iteration theorem. Recall that for $z_1 \le z_2 \le z_3$

$$\boldsymbol{Q}(z_1, z_3) = \boldsymbol{Q}(z_1, z_2)\boldsymbol{Q}(z_2, z_3)$$

Substituting equation (51) in the above equation, we have

$$\begin{bmatrix} \boldsymbol{T}_u^{13} - \boldsymbol{R}_D^{31}\left(\boldsymbol{T}_D^{31}\right)^{-1}\boldsymbol{R}_u^{13} & \boldsymbol{R}_D^{31}\left(\boldsymbol{T}_D^{31}\right)^{-1} \\ -\left(\boldsymbol{T}_D^{31}\right)^{-1}\boldsymbol{R}_u^{13} & \left(\boldsymbol{T}_D^{31}\right)^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_u^{12} - \boldsymbol{R}_D^{21}\left(\boldsymbol{T}_D^{21}\right)^{-1}\boldsymbol{R}_u^{12} & \boldsymbol{R}_D^{21}\left(\boldsymbol{T}_D^{21}\right)^{-1} \\ -\left(\boldsymbol{T}_D^{21}\right)^{-1}\boldsymbol{R}_u^{12} & \left(\boldsymbol{T}_D^{21}\right)^{-1} \end{bmatrix}$$
$$\begin{bmatrix} \boldsymbol{T}_u^{23} - \boldsymbol{R}_D^{32}\left(\boldsymbol{T}_D^{32}\right)^{-1}\boldsymbol{R}_u^{23} & \boldsymbol{R}_D^{32}\left(\boldsymbol{T}_D^{32}\right)^{-1} \\ -\left(\boldsymbol{T}_D^{32}\right)^{-1}\boldsymbol{R}_u^{23} & \left(\boldsymbol{T}_D^{32}\right)^{-1} \end{bmatrix} \tag{52}$$

Now doing the multiplcation and solving for $\boldsymbol{R}_D^{31}$, $\boldsymbol{T}_D^{31}$, $\boldsymbol{R}_u^{13}$ and $\boldsymbol{T}_u^{13}$ in terms of $\boldsymbol{R}_D^{21}$ etc. we get,

$$\begin{aligned} \boldsymbol{R}_D^{31} &= \boldsymbol{R}_D^{21} + \boldsymbol{T}_u^{12}\boldsymbol{R}_D^{32}\left[\boldsymbol{I} - \boldsymbol{R}_u^{12}\boldsymbol{R}_D^{32}\right]^{1}\boldsymbol{T}_D^{21} \\ \boldsymbol{T}_D^{31} &= \boldsymbol{T}_D^{32}\left[\boldsymbol{I} - \boldsymbol{R}_u^{12}\boldsymbol{R}_D^{32}\right]^{1}\boldsymbol{T}_D^{21} \\ \boldsymbol{R}_u^{13} &= \boldsymbol{R}_u^{23} + \boldsymbol{T}_D\boldsymbol{R}_u^{12}\left[\boldsymbol{I} - \boldsymbol{R}_D^{32}\boldsymbol{R}_u^{12}\right]^{-1}\boldsymbol{T}_u^{23} \\ \boldsymbol{T}_u^{13} &= \boldsymbol{T}_u^{12}\left[\boldsymbol{I} - \boldsymbol{R}_D^{32}\boldsymbol{R}_u^{12}\right]^{1}\boldsymbol{T}_u^{23} \end{aligned} \tag{53}$$

These are the iteration equations. Since $z_1 \le z_2 \le z_3$ is arbitrary, it follows that we can compute the R/T coefficients for an inhomogeneous zone by dividing that zone into a number of layers and iterating them layer by layer.

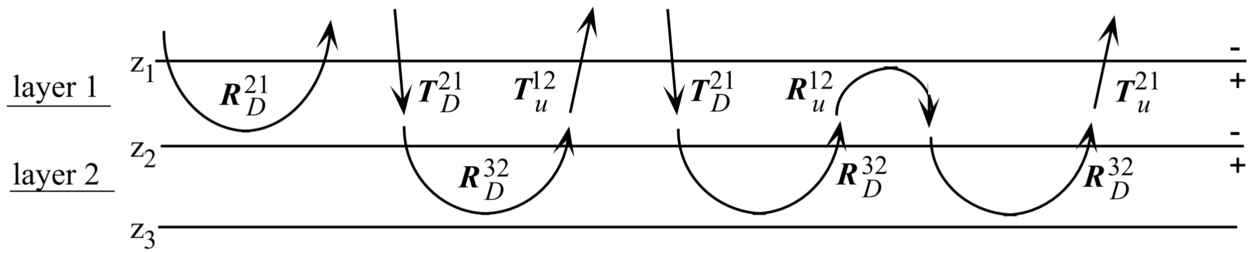
Interpretation of iteration equations

Use the identity

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}\boldsymbol{A} + \ldots$$

Now

$$\boldsymbol{R}_D^{31} = \boldsymbol{R}_D^{21} + \boldsymbol{T}_u^{12}\boldsymbol{R}_D^{32}\left[\boldsymbol{I} - \boldsymbol{R}_u^{12}\boldsymbol{R}_D^{32}\right]^{1}\boldsymbol{T}_D^{21}$$
$$\boldsymbol{R}_D^{31} = \boldsymbol{R}_D^{21} + \boldsymbol{T}_u^{12}\boldsymbol{R}_D^{32}\boldsymbol{T}_D^{21} + \boldsymbol{T}_u^{12}\boldsymbol{R}_D^{32}\boldsymbol{R}_u^{12}\boldsymbol{R}_D^{32}\cdot\boldsymbol{T}_D^{21} + \boldsymbol{T}_u^{12}\boldsymbol{R}_D^{32}\boldsymbol{R}_u^{12}\boldsymbol{R}_D^{32}\boldsymbol{R}_u^{12}\boldsymbol{R}_D^{32}\boldsymbol{T}_D^{21} + \ldots$$


layer 1
layer 2
z_1
z_2
z_3
R_D^{21}
T_D^{21}
T_u^{12}
R_D^{32}
T_D^{21}
R_u^{12}
R_D^{32}
R_D^{32}
T_u^{21}
-
+
-
+